Chapter 3

Calculus in $\mathbb{R}$

3.1 Differentiation

Definition 3.1. *Suppose $U \subset \mathbb{R}$ is open. A function $f\colon U \to \mathbb{R}$ is* ***differentiable at $x \in U$*** *if there exists a number m such that*

$$\lim_{y \to 0} \left[\frac{|f(x+y) - f(x) - my|}{|y|} \right] = 0.$$

If f is differentiable at each $x \in U$ then we say that ***f is differentiable on U****.*

Lemma 3.2. *Suppose $U \subset \mathbb{R}$ is open and $f\colon U \to \mathbb{R}$ is differentiable on U. Then for each $x \in U$ there exists a unique number m such that*

$$\lim_{y \to 0} \left[\frac{|f(x+y) - f(x) - my|}{|y|} \right] = 0.$$

Thus $x \mapsto m$ is a function $U \to \mathbb{R}$, which we denote f' and call the ***derivative of f****.*

Lemma 3.3. *Suppose $U \subset \mathbb{R}$ is open and $f\colon U \to \mathbb{R}$ is differentiable on U. Then for each $x \in U$ we have*

$$f'(x) = \lim_{y \to 0} \left[\frac{f(x+y) - f(x)}{y} \right].$$

Proposition 3.4 (Properties of differentiation)**.** *Suppose $U \subset \mathbb{R}$ is open and $f, g\colon U \to \mathbb{R}$ are differentiable on U. Then*

1. *$f\colon U \to \mathbb{R}$ is continuous;*

2. *$f+g$ is differentiable on U with $(f+g)' = f' + g'$;*

3. *for each $a \in \mathbb{R}$ the function af is differentiable on U with $(af)' = af'$;*

4. *the function fg is differentiable on U with $(fg)' = f'g + fg'$; and*

5. *if g is differentiable on $f(U)$ then $f \circ g$ is differentiable with $(f \circ g)' = (f' \circ g)g'$.*

Definition 3.5. *Let $U \subset \mathbb{R}$ be open, and let $f : U \to \mathbb{R}$ be a function.*

1. *A point $x \in U$ is a **local minimizer for f** if there exists $r > 0$ such that $f(x) \leq f(y)$ for all $y \in U \cap B_r(x) \setminus \{x\}$. If the inequality is strict, then x is a **strict local minimizer**.*

2. *A point $x \in U$ is a **local maximizer for f** if there exists $r > 0$ such that $f(x) \geq f(y)$ for all $y \in U \cap B_r(x) \setminus \{x\}$. If the inequality is strict, then x is a **strict local maximizer**.*

*If x is either a (strict) local minimizer or a (strict) local maximizer, then we say that x is an **(isolated) extremizer**.*

Lemma 3.6. *Suppose $U \subset \mathbb{R}$ is open and that $f : U \to \mathbb{R}$ is differentiable. If $x \in U$ is an extremizer of f, then $f'(x) = 0$.*

Theorem 3.7 (Mean value theorem)**.** *Let $a, b \in \mathbb{R}$. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that f is differentiable on (a, b). Then there exists $x_* \in (a, b)$ such that*

$$f'(x_*) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 3.8 (Uniqueness of anti-derivatives)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and is differentiable on (a, b). Then f is constant if and only if $f' = 0$ on (a, b).*

Definition 3.9. *Suppose $U \subset \mathbb{R}$ is open and $f : U \to \mathbb{R}$ is differentiable on U. We say that f is **continuously differentiable** if the function $f' : U \to \mathbb{R}$ is continuous. The set of continuously differentiable functions $U \to \mathbb{R}$ is denoted by $C^1(U; \mathbb{R})$ or simply by $C^1(U)$.*

Lemma 3.10 (Local monotonicity lemma)**.** *Let $U \subset \mathbb{R}$ is open and let $f \in C^1(U)$. Suppose $x_* \in U$ is such that $f'(x_*) > 0$. Then there exists $r > 0$ such that if $x, y \in B_r(x_*)$ with $x < y$ then $f(x) < f(y)$.*

Definition 3.11. *If the derivative f' of function f is itself differentiable, then we say that f is* ***twice differentiable****. The derivative $(f')'$ is called the* ***second derivative*** *and is denoted f''.*

More generally, we denote the ***k^{th} derivative*** *of f by $f^{(k)}$. The collection of functions $f\colon U \to \mathbb{R}$ for which all derivatives up to order k are continuous is denoted $C^k(U;\mathbb{R})$ or simply $C^k(U)$.*

Proposition 3.12 (Second derivative test)**.** *Suppose that $U \subset \mathbb{R}$ is open, that $f \in C^2(U)$, and $x_* \in U$ is such that $f'(x_*) = 0$. Then*

- *if $f''(x_*) > 0$ then x_* is a strict minimizer, and*
- *if $f''(x_*) < 0$ then x_* is a strict maximizer.*

Theorem 3.13 (Taylor expansion)**.** *Suppose $f \in C^{n+1}(B_r(x_*))$. Then for each $x \in B_r(x_*)$ there exists η between x and x_* such that*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_*)\,(x - x_*)^k + \frac{1}{(n+1)!} f^{(n+1)}(\eta)\,(x - x_*)^{n+1}.$$

3.2 Integration

Definition 3.14. *Let $[a, b] \subset \mathbb{R}$.*

1. *A* ***partition*** *of $[a, b]$ is a finite set $P = \{x_i\}_{i=0}^n \subset [a, b]$ such that $a = x_0 \le x_1 \le \cdots \le x_n = b$.*
2. *The* ***diameter*** *of partition P is defined by $\|P\| = \max_i |x_i - x_{i-1}|$.*
3. *A* ***refinement P'*** *of partition P is itself a partition of $[a, b]$ and must have the property that $P \subset P'$.*
4. *If P and Q are partitions of $[a, b]$ we call $P \cup Q$ their* ***common refinement****.*

The collection of all partitions of $[a, b]$ is denoted $\mathscr{P}([a, b])$.

Definition 3.15. *Suppose $f\colon [a, b] \to \mathbb{R}$ is a bounded function and $P = \{x_i\}_{i=0}^n$ is a partition of $[a, b]$.*

1. *The* ***upper Darboux sum of f with respect to P****, which we denote $U(P, f)$, is the sum*

$$U(P, f) = \sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

2. *The **lower Darboux sum of f with respect to P**, which we denote $L(P,f)$, is the sum*

$$L(P,f)=\sum_{i=1}^{n}\left(\inf_{[x_{i-1},x_i]} f(x)\right)(x_i-x_{i-1})$$

Lemma 3.16. *Suppose P is a partition of $[a,b]\subset\mathbb{R}$ and that $f\colon[a,b]\to\mathbb{R}$ is bounded.*

1. *There exists constant M such that*

$$-M(b-a)\leq L(P,f)\leq U(P,f)\leq M(b-a).$$

2. *If P' is a refinement of P, then*

$$L(P,f)\leq L(P',f)\leq U(P',f)\leq U(P,f).$$

3. *If Q is another partition of $[a,b]$, then*

$$L(P,f)\leq U(Q,f).$$

Definition 3.17. *Suppose $f\colon[a,b]\to\mathbb{R}$ is bounded.*

1. *The **upper Darboux integral** of f on $[a,b]$ is defined by*

$$\overline{\int_a^b} f(x)\,dx=\inf\{U(P,f)\mid P\in\mathscr{P}([a,b])\}$$

2. *The **lower Darboux integral** of f on $[a,b]$ is defined by*

$$\underline{\int_a^b} f(x)\,dx=\sup\{L(P,f)\mid P\in\mathscr{P}([a,b])\}$$

3. *If the upper and lower Darboux integrals are the same, then we say that f is **integrable** and define the Darboux integral of f on $[a,b]$ by*

$$\int_a^b f(x)\,dx=\overline{\int_a^b} f(x)\,dx=\underline{\int_a^b} f(x)\,dx$$

Lemma 3.18. *Let $f\colon[a,b]\to\mathbb{R}$ be a bounded function. Then*

$$\underline{\int_a^b} f(x)\,dx\leq\overline{\int_a^b} f(x)\,dx$$

Theorem 3.19 (Darboux integrability criterion)**.** *A bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Example 3.20. *The following functions are integrable on any domain $[a,b]$:*

- *the function $f(x) = c$, where c is any constant, and*
- *the function $f(x) = x$.*

The following function is not integrable on any domain $[a,b]$:

- *the function*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Proposition 3.21. *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Then f is integrable.*

Definition 3.22. *A function $f\colon [a,b] \to \mathbb{R}$ is **monotone** if either*

- *$f(x) \le f(y)$ whenever $x \le y$, in which case f is called **monotone increasing**, or*
- *$f(x) \ge f(y)$ whenever $x \le y$, in which case f is called **monotone decreasing**.*

Proposition 3.23. *Suppose $f\colon [a,b] \to \mathbb{R}$ is monotone. Then f is integrable.*

Theorem 3.24 (Darboux convergence criterion)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $I \in \mathbb{R}$. Then the following are equivalent.*

1. *The function f is integrable and $\int_a^b f(x)\,dx = I$.*
2. *Whenever $\{P_k\}_{k=1}^{\infty}$ is a sequence of partitions of $[a,b]$ such that $\|P_k\| \to 0$ then*

$$L(P_k,f) \to I \qquad \text{and} \qquad U(P_k,f) \to I.$$

Definition 3.25. *Let $[a,b] \subset \mathbb{R}$.*

1. *A **pointed partition** $\widehat{P}$ of interval $[a,b]$ consists of a partition $P = \{x_i\}_{i=0}^{n}$ of $[a,b]$ together with a set of points $C = \{c_i\}_{i=1}^{n}$ such that $x_{i-1} \le c_i \le x_i$ for each $i = 1, \dots, n$.*

2. *Suppose* $\widehat{P} = (P, C)$ *is a pointed partition of* $[a, b]$*. The* ***diameter*** *of* $\widehat{P}$*, which we denote* $\|\widehat{P}\|$*, is defined as the diameter of the partition* P*.*

3. *Suppose* $\widehat{P}$ *is a pointed partition of* $[a, b]$*. The* ***Riemann sum*** *of* f *with respect to* $\widehat{P}$*, which we denote* $R(\widehat{P}, f)$*, is the sum*

$$R(\widehat{P}, f) = \sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1})\,.$$

The collection of all pointed partitions of $[a, b]$ *is denoted* $\widehat{\mathscr{P}}([a, b])$*.*

Lemma 3.26. *Suppose* $\widehat{P} = (P, C)$ *is a pointed partition of* $[a, b]$ *and* $f\colon [a, b] \to \mathbb{R}$ *is a bounded function. Then*

$$L(P, f) \leq R(\widehat{P}, f) \leq U(P, f).$$

Theorem 3.27 (Riemann convergence criterion)**.** *Suppose* $f\colon [a, b] \to \mathbb{R}$ *and* $I \in \mathbb{R}$*. Then the following are equivalent.*

1. f *is integrable and* $\int_a^b f(x)\,dx = I$*.*

2. *Whenever* $\widehat{P}_k$ *is a sequence of pointed partitions of* $[a, b]$ *such that* $\|\widehat{P}_k\| \to 0$ *then* $R(\widehat{P}_k, f) \to I$*.*

Remark 3.28. *The limit*

$$\lim_{\|\widehat{P}_k\| \to 0} R(\widehat{P}_k, f)$$

is called the Riemann integral of f*.*

Definition 3.29. *Suppose* $f\colon [a, b] \to \mathbb{R}$ *is integrable. We define*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proposition 3.30 (Properties of integrals)**.**

1. *(Linearity of integration I) If* f, g *are integrable on* $[a, b]$*, then so is* $f + g$ *and*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *(Linearity of integration II) If* f *is integrable on* $[a,b]$ *and* $c \in \mathbb{R}$*, then* cf *is Darboux integrable and*

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$$

3. *(Monotonicity of integration) If* f, g *are integrable on* $[a,b]$ *with* $f(x) \leq g(x)$ *for each* $x \in [a,b]$ *then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

4. *(Additivity of intervals) Function* f *is integrable on intervals* $[a,b]$ *and* $[b,c]$ *if and only if it is Darboux integrable on* $[a,c]$*, in which case*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Lemma 3.31. *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is integrable, and* $g\colon \mathbb{R} \to \mathbb{R}$ *is continuous. Then* $g \circ f\colon [a,b] \to \mathbb{R}$ *is integrable.*

Corollary 3.32. *If* f *is integrable on* $[a,b]$ *then so is* $|f|$*, with*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proposition 3.33 (Mean value theorem for integrals)**.** *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is continuous. Then there exists* $c \in [a,b]$ *such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Theorem 3.34 (Fundamental theorem of calculus)**.** *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is integrable on any closed interval contained in* $[a,b]$*.*

1. *The function* $F\colon [a,b] \to \mathbb{R}$ *defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous. Furthermore, if f *is continuous then* F *is differentiable with* $F' = f$*.*

2. *Suppose* $F\colon [a,b] \to \mathbb{R}$ *is differentiable on* (a,b) *with* $F' = f$*. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proposition 3.35 (Integration by parts)**.** *Suppose* $f, g\colon [a,b] \to \mathbb{R}$ *are continuous functions that are continuously differentiable on* (a,b)*. Then*

$$\int_a^b f(x)\,g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx.$$

Proposition 3.36 (Change of variables)**.** *Suppose* $g\colon [a,b] \to \mathbb{R}$ *is continuous, is differentiable on* (a,b)*, and that* g' *extends to a continuous function on* $[a,b]$*. Suppose also that* f *is continuous on* $g([a,b])$*. Then*

$$\int_a^b (f \circ g)(x)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Proposition 3.37. *Suppose that for each* $n = 1, 2, 3, \ldots$ *we have an integrable function* $f_n\colon [a,b] \to \mathbb{R}$ *and suppose that* f_n *converges to* $f\colon [a,b] \to \mathbb{R}$ *with respect to the* $L^\infty([a,b])$ *norm. Then* f *is integrable and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$