Chapter 3

Calculus in $\mathbb{R}$

3.1 Differentiation

Definition 3.1. *Suppose $U \subset \mathbb{R}$ is open. A function $f: U \to \mathbb{R}$ is* ***differentiable at $x \in U$*** *if there exists a number m such that*

$$\lim_{y \to 0} \left[\frac{|f(x+y) - f(x) - my|}{|y|} \right] = 0.$$

If f is differentiable at each $x \in U$ then we say that ***f is differentiable on U****.*

Lemma 3.2. *Suppose $U \subset \mathbb{R}$ is open and $f: U \to \mathbb{R}$ is differentiable on U. Then for each $x \in U$ there exists a unique number m such that*

$$\lim_{y \to 0} \left[\frac{|f(x+y) - f(x) - my|}{|y|} \right] = 0.$$

Thus $x \mapsto m$ is a function $U \to \mathbb{R}$, which we denote f' and call the ***derivative of f****.*

Proof. The existence is by hypothesis and the uniqueness follows from a straightforward contradiction argument. □

Lemma 3.3. *Suppose $U \subset \mathbb{R}$ is open and $f: U \to \mathbb{R}$ is differentiable on U. Then for each $x \in U$ we have*

$$f'(x) = \lim_{y \to 0} \left[\frac{f(x+y) - f(x)}{y} \right].$$

Proof. This follows immediately from the definition of $f'(x)$. □

Proposition 3.4 (Properties of differentiation)**.** *Suppose* $U \subset \mathbb{R}$ *is open and* $f, g \colon U \to \mathbb{R}$ *are differentiable on* U*. Then*

1. $f \colon U \to \mathbb{R}$ *is continuous;*
2. $f + g$ *is differentiable on* U *with* $(f+g)' = f' + g'$*;*
3. *for each* $a \in \mathbb{R}$ *the function* af *is differentiable on* U *with* $(af)' = af'$*;*
4. *the function* fg *is differentiable on* U *with* $(fg)' = f'g + fg'$*; and*
5. *if* g *is differentiable on* $f(U)$ *then* $f \circ g$ *is differentiable with* $(f \circ g)' = (f' \circ g)g'$*.*

Proof. These follow directly from the definition. □

Definition 3.5. *Let* $U \subset \mathbb{R}$ *be open, and let* $f \colon U \to \mathbb{R}$ *be a function.*

1. *A point* $x \in U$ *is a* ***local minimizer for f*** *if there exists* $r > 0$ *such that* $f(x) \leq f(y)$ *for all* $y \in U \cap B_r(x) \setminus \{x\}$*. If the inequality is strict, then* x *is a* ***strict local minimizer****.*
2. *A point* $x \in U$ *is a* ***local maximizer for f*** *if there exists* $r > 0$ *such that* $f(x) \geq f(y)$ *for all* $y \in U \cap B_r(x) \setminus \{x\}$*. If the inequality is strict, then* x *is a* ***strict local maximizer****.*

If x *is either a (strict) local minimizer or a (strict) local maximizer, then we say that* x *is an* ***(isolated) extremizer****.*

Lemma 3.6. *Suppose* $U \subset \mathbb{R}$ *is open and that* $f \colon U \to \mathbb{R}$ *is differentiable. If* $x \in U$ *is an extremizer of* f*, then* $f'(x) = 0$*.*

Proof. Without loss of generality, x is a local minimizer for f. Notice that

$$f'(x) = \lim_{n\to\infty} \frac{f(x+1/n) - f(x)}{1/n} \quad \text{and} \quad f'(x) = \lim_{n\to\infty} \frac{f(x-1/n) - f(x)}{-1/n}.$$

But for n sufficiently large we have

$$\frac{f(x+1/n) - f(x)}{1/n} \geq 0 \quad \text{and} \quad \frac{f(x-1/n) - f(x)}{-1/n} \leq 0.$$

Thus $f'(x) \geq 0$ and $f'(x) \leq 0$, which implies $f'(x) = 0$. □

Theorem 3.7 (Mean value theorem)**.** *Let $a, b \in \mathbb{R}$. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous and that f is differentiable on (a,b). Then there exists $x_* \in (a,b)$ such that*

$$f'(x_*) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider the function $g\colon [a,b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Notice that g is continuous on $[a,b]$, is differentiable on (a,b), and satisfies

$$g(a) = 0 \quad \text{and} \quad g(b) = 0.$$

If $g(x) = 0$ for all $x \in [a,b]$, then simply choose $x_* = (a+b)/2$. Otherwise, there exists some $x \in (a,b)$ such that $g(x) \neq 0$. Without loss of generality, assume that $g(x) > 0$. (Otherwise, replace g by $-g$.)

Since $[a,b]$ is compact and g is continuous, there exists $x_* \in [a,b]$ such that $g(x_*) = \sup\{g(x) \mid x \in [a,b]\}$. Clearly $x_* \neq a$ and $x_* \neq b$. Thus g has an extremizer on (a,b), where it is differentiable. Consequently, $g'(x_*) = 0$. The desired identity follows by computing the derivative of g. □

Corollary 3.8 (Uniqueness of anti-derivatives)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and is differentiable on (a,b). Then f is constant if and only if $f' = 0$ on (a,b).*

Proof. If f is constant, then the definition of the derivative implies that $f' = 0$. Suppose, then, that $f' = 0$. Let $c, d \in [a,b]$ such that $c < d$. Applying the mean value theorem to the interval $[c,d]$ we find that there is some $x_* \in (c,d)$ such that

$$\frac{f(d) - f(c)}{d - c} = f'(x_*) = 0.$$

This easily implies $f(d) = f(c)$. □

Definition 3.9. *Suppose $U \subset \mathbb{R}$ is open and $f\colon U \to \mathbb{R}$ is differentiable on U. We say that f is* ***continuously differentiable*** *if the function $f'\colon U \to \mathbb{R}$ is continuous. The set of continuously differentiable functions $U \to \mathbb{R}$ is denoted by $C^1(U;\mathbb{R})$ or simply by $C^1(U)$.*

Lemma 3.10 (Local monotonicity lemma)**.** *Let $U \subset \mathbb{R}$ is open and let $f \in C^1(U)$. Suppose $x_* \in U$ is such that $f'(x_*) > 0$. Then there exists $r > 0$ such that if $x, y \in B_r(x_*)$ with $x < y$ then $f(x) < f(y)$.*

Proof. The continuity of f' implies that there exists $r > 0$ such that $|x - x_*| < r$ implies $f'(x) \geq \frac{1}{2} f'(x_*) > 0$. Suppose now that there exists $x, y \in B_r(x_*)$ with $x < y$ and $f(x) \geq f(y)$. Then the mean value theorem would imply that there exists $y_* \in (x, y)$ such that $f'(y_*) \leq 0$, which is a contradiction. □

Definition 3.11. *If the derivative f' of function f is itself differentiable, then we say that f is **twice differentiable**. The derivative $(f')'$ is called the **second derivative** and is denoted f''.*

*More generally, we denote the **k^{th} derivative** of f by $f^{(k)}$. The collection of functions $f\colon U \to \mathbb{R}$ for which all derivatives up to order k are continuous is denoted $C^k(U;\mathbb{R})$ or simply $C^k(U)$.*

Proposition 3.12 (Second derivative test)**.** *Suppose that $U \subset \mathbb{R}$ is open, that $f \in C^2(U)$, and $x_* \in U$ is such that $f'(x_*) = 0$. Then*

- *if $f''(x_*) > 0$ then x_* is a strict minimizer, and*
- *if $f''(x_*) < 0$ then x_* is a strict maximizer.*

Proof. Suppose $f \in C^2(U)$ and that $x_* \in U$ with $f'(x_*) = 0$ and $f''(x_*) > 0$. Applying the monotonicity lemma to f' we obtain the existence of $r > 0$ such that whenever $x, y \in B_r(x_*)$ with $x < y$ then $f'(x) < f'(y)$.

Suppose that $x \in B_r(x_*)$ is such that $f(x) \leq f(x_*)$. If $x > x_*$ then the mean value theorem implies the existence of $y_* \in (x_*, x)$ such that $f'(y_*) \leq 0$, which is a contradiction. A similar contradiction arises if $x < x_*$.

The proof in the case that $f''(x_*) < 0$ follows by replacing f by $-f$. □

Theorem 3.13 (Taylor expansion)**.** *Suppose $f \in C^{n+1}(B_r(x_*))$. Then for each $x \in B_r(x_*)$ there exists η between x and x_* such that*

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_*)\,(x - x_*)^k + \frac{1}{(n+1)!} f^{(n+1)}(\eta)\,(x - x_*)^{n+1}.$$

Proof. Without loss of generality we may assume $x > x_*$ and take $x_* = 0$.

The intermediate value theorem implies that we may find $y \in \mathbb{R}$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^k + \frac{1}{(n+1)!}\, x^{n+1} y.$$

Define $g\colon [0, x] \to \mathbb{R}$ by

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)\,(x - t)^k - \frac{1}{(n+1)!}\,(x - t)^{n+1} y.$$

Note that $g \in C^1((0,x))$.

Since $g(0) = 0$ and $g(x) = 0$ the mean value theorem implies that there exists $\eta \in (0,x)$ such that $g'(\eta) = 0$. Direct computation shows that $g'(\eta) = 0$ implies that $y = f^{(n+1)}(\eta)$ as desired. □

3.2 Integration

Definition 3.14. *Let* $[a,b] \subset \mathbb{R}$.

1. *A* ***partition*** *of* $[a,b]$ *is a finite set* $P = \{x_i\}_{i=0}^n \subset [a,b]$ *such that* $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$.
2. *The* ***diameter*** *of partition* P *is defined by* $\|P\| = \max_i |x_i - x_{i-1}|$.
3. *A* ***refinement*** $\boldsymbol{P'}$ *of partition* P *is itself a partition of* $[a,b]$ *and must have the property that* $P \subset P'$.
4. *If* P *and* Q *are partitions of* $[a,b]$ *we call* $P \cup Q$ *their* ***common refinement***.

The collection of all partitions of $[a,b]$ *is denoted* $\mathscr{P}([a,b])$.

Definition 3.15. *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is a bounded function and* $P = \{x_i\}_{i=0}^n$ *is a partition of* $[a,b]$.

1. *The* ***upper Darboux sum of*** $\boldsymbol{f}$ ***with respect to*** $\boldsymbol{P}$, *which we denote* $U(P,f)$, *is the sum*
$$U(P,f) = \sum_{i=1}^n \left(\sup_{[x_{i-1},x_i]} f(x) \right) (x_i - x_{i-1})$$
2. *The* ***lower Darboux sum of*** $\boldsymbol{f}$ ***with respect to*** $\boldsymbol{P}$, *which we denote* $L(P,f)$, *is the sum*
$$L(P,f) = \sum_{i=1}^n \left(\inf_{[x_{i-1},x_i]} f(x) \right) (x_i - x_{i-1})$$

Lemma 3.16. *Suppose* P *is a partition of* $[a,b] \subset \mathbb{R}$ *and that* $f\colon [a,b] \to \mathbb{R}$ *is bounded.*

1. *There exists constant* M *such that*
$$-M(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

2. *If P' is a refinement of P, then*

$$L(P,f) \leq L(P',f) \leq U(P',f) \leq U(P,f).$$

3. *If Q is another partition of $[a,b]$, then*

$$L(P,f) \leq U(Q,f).$$

Proof. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$.

1. We have
$$U(P,f) \leq M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b-a);$$
similarly $L(P,f) \geq -M$. The remaining inequality follows from the infimum of any set being no greater than the supremum.

2. Let $P = \{x_i\}_{i=0}^{n}$ and $P' = \{x_i'\}_{i=0}^{n'}$. For each $x_i' \in P' \setminus P$ there exists x_k such that $x_{k-1} < x_i' < x_k$. Since
$$\inf_{[x_{k-1},x_k]} f(x) \leq \max\{\inf_{[x_{k-1},x_i']} f(x), \inf_{[x_i',x_k]} f(x)\}$$
and $x_k - x_{k-1} = x_k - x_i' + x_i' - x_{k-1}$ it follows immediately from the definition that $L(P,f) \leq L(P',f)$.

   The argument that $U(P',f) \leq U(P,f)$ follows similarly.

3. Let $P' = P \cup Q$. Then $L(P,f) \leq L(P',f) \leq U(P',f) \leq U(Q,f)$. □

Definition 3.17. *Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded.*

1. *The* ***upper Darboux integral*** *of f on $[a,b]$ is defined by*
$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \mathscr{P}([a,b])\}$$

2. *The* ***lower Darboux integral*** *of f on $[a,b]$ is defined by*
$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \in \mathscr{P}([a,b])\}$$

3. *If the upper and lower Darboux integrals are the same, then we say that f is* ***integrable*** *and define the Darboux integral of f on $[a,b]$ by*
$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Lemma 3.18. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then*

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

Proof. This follows from the fact that $L(P,f) \le U(P,f)$ for all $P \in \mathscr{P}([a,b])$. □

Theorem 3.19 (Darboux integrability criterion)**.** *A bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof. Suppose f is integrable and let $\varepsilon > 0$. There exists $Q_+, Q_- \in \mathscr{P}([a,b])$ such that

$$\left|U(Q_+,f) - \int_a^b f(x)\,dx\right| + \left|L(Q_-,f) - \int_a^b f(x)\,dx\right| < \varepsilon.$$

Let P be the common refinement of Q_+ and Q_-. Then

$$U(P,f) - L(P,f) \le U(Q_+,f) - L(Q_-,f) \le \varepsilon.$$

For the converse, let $\{P_n\}_{n=0}^{\infty} \subset \mathscr{P}([a,b])$ such that

$$U(P_n,f) - L(P_n,f) < \frac{1}{n}.$$

Since f is bounded, the sequence $U(P_n,f)$ is a bounded sequence of real numbers. Thus there exists a subsequence $\{P_{n_k}\}_{k=1}^{\infty}$ such that $U(P_{n_k})$ converges to $I \in \mathbb{R}$.

Let $\varepsilon > 0$ and choose $K \in \mathbb{N}$ such that $k \ge K$ implies $1/n_k < \varepsilon/2$ and $|U(P_{n_k},f) - I| < \varepsilon/2$. Then for $k \ge K$ we have

$$|L(P_{n_k},f) - I| \le |L(P_{n_k},f) - U(P_{n_k},f)| + |U(P_{n_k},f) - I| < \varepsilon,$$

which implies $L(P_{n_k},f) \to I$.

However, for all $k \in \mathbb{N}$ we have

$$L(P_{n_k},f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P_{n_k},f).$$

It follows immediately that the upper and lower integrals must be equal, and thus f is integrable. □

Example 3.20. *The following functions are integrable on any domain $[a, b]$:*

- *the function $f(x) = c$, where c is any constant, and*
- *the function $f(x) = x$.*

The following function is not integrable on any domain $[a, b]$:

- *the function*
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Proposition 3.21. *Suppose $f \colon [a, b] \to \mathbb{R}$ is continuous. Then f is integrable.*

Proof. Let $\varepsilon > 0$. Since $[a, b]$ is compact, the function f is uniformly continuous. Thus there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y) < \varepsilon/2(b - a)$. Let $P = \{x_i\}_{i=0}^n$ be a partition of $[a, b]$ such that $|x_i - x_{i-1}| < \delta$ for all $i \in \mathbb{N}$.

The continuity of f further implies that for each $i \in \mathbb{N}$ there exists $x_i^+, x_i^- \in [x_{i-1}, x_i]$ such that

$$\sup_{[x_{i-1}, x_i]} = f(x_i^+) \quad \text{and} \quad \inf_{[x_{i-1}, x_i]} = f(x_i^-).$$

Thus

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^n \left(f(x_i^+) - f(x_i^-)\right)(x_i - x_{i-1}) \\ &\leq \frac{\varepsilon}{2(b - a)} \sum_{i=1}^n (x_i - x_{i-1}) = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

The Darboux integrability criterion implies f is integrable. □

Definition 3.22. *A function $f \colon [a, b] \to \mathbb{R}$ is* ***monotone*** *if either*

- *$f(x) \leq f(y)$ whenever $x \leq y$, in which case f is called* ***monotone increasing****, or*
- *$f(x) \geq f(y)$ whenever $x \leq y$, in which case f is called* ***monotone decreasing****.*

Proposition 3.23. *Suppose $f \colon [a, b] \to \mathbb{R}$ is monotone. Then f is integrable.*

Proof. We give the proof only in the case that f is monotone increasing; the proof in the decreasing case is analogous.

Suppose $P = \{x_k\}_{k=0}^n$ is a partition of $[a, b]$. Since f is monotone increasing we have

$$U(P, f) = \sum_{k=1}^{n} f(x_k)\,(x_k - x_{k-1})$$
$$L(P, f) = \sum_{k=0}^{n} f(x_{k-1})\,(x_k - x_{k-1}).$$

Thus

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\,(x_k - x_{k-1}) \\ &\leq \|P\| \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) \\ &= \|P\|(f(b) - f(a)). \end{aligned}$$

Taking a sequence of partitions such that the diameter tends to zero, we see that f is integrable by the Darboux integrability criterion. □

Theorem 3.24 (Darboux convergence criterion)**.** *Suppose* $f: [a, b] \to \mathbb{R}$ *is bounded and* $I \in \mathbb{R}$*. Then the following are equivalent.*

1. *The function* f *is integrable and* $\int_a^b f(x)\,dx = I$*.*

2. *Whenever* $\{P_k\}_{k=1}^\infty$ *is a sequence of partitions of* $[a, b]$ *such that* $\|P_k\| \to 0$ *then*
$$L(P_k, f) \to I \qquad \text{and} \qquad U(P_k, f) \to I.$$

Proof. Choose $M > 0$ such that $|f(x)| < M$ for all $x \in [a, b]$.

Suppose $f: [a, b] \to \mathbb{R}$ is integrable; set $I = \int_a^b f(x)\,dx$. Let $\{P_k\}_{k=1}^\infty$ be a sequence of partitions of $[a, b]$ such that $\|P_k\| \to 0$ and fix $\varepsilon > 0$.

By the Darboux integrability criterion, there exists partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2}.$$

Consider now the partition P_k, which we write $P_k = \{x_0^k, \ldots, x_{m_k}^k\}$. Notice that there are at most n subintervals $[x_j^k, x_{j+1}^k]$ containing one of the points x_j of the partition P. Let $J \subset \{0, \dots, m_k - 1\}$ be the collection of indices j such that $[x_j^k, x_{j+1}^k]$ has this property. Thus

$$U(P_k, f) - L(P_k, f) = \sum_{j \in J} \left[\sup_{[x_j^k, x_{j+1}^k]} f - \inf_{[x_j^k, x_{j+1}^k]} f \right] (x_{j+1}^k - x_j^k) + \sum_{j \notin J} \left[\sup_{[x_j^k, x_{j+1}^k]} f - \inf_{[x_j^k, x_{j+1}^k]} f \right] (x_{j+1}^k - x_j^k)$$

Notice that

$$\sum_{j \in J} \left[\sup_{[x_j^k, x_{j+1}^k]} f - \inf_{[x_j^k, x_{j+1}^k]} f \right] (x_{j+1}^k - x_j^k) \leq Mn\|P_k\|.$$

Furthermore, there exists $N \in \mathbb{N}$ such that $k \geq N$ implies $Mn\|P_k\| < \varepsilon/2$.

For each $j \notin J$ the interval $[x_j^k, x_{j+1}^k]$ is contained in some interval arising from the partition P. Thus

$$\sum_{j \notin J} \left[\sup_{[x_j^k, x_{j+1}^k]} f - \inf_{[x_j^k, x_{j+1}^k]} f \right] (x_{j+1}^k - x_j^k) \leq U(P, f) - L(P, f) < \frac{\varepsilon}{2}.$$

Consequently, we find that for $k \geq N$ we have

$$U(P_k, f) - L(P_k, f) < \varepsilon.$$

Since $L(P_k, f) \leq I \leq U(P_k, f)$, this implies that $k \geq N$ we have

$$|L(P_k, f) - I| < \varepsilon \quad \text{and} \quad |U(P_k, f) - I| < \varepsilon.$$

Thus $U(P_k, f) \to I$ and $L(_k, f) \to I$.

We now prove that the second statement implies the first. For each $n \in \mathbb{N}$, let P_n be the partition that divides $[a, b]$ in to n equally-sized subintervals. Clearly $\|P_n\| \to 0$, and thus $L(P_n, f) \to I$ and $U(P_n, f) \to I$. In particular, for any $\varepsilon > 0$ we may choose $N \in \mathbb{N}$ such that $U(P_N, f) - L(P_N, f) < \varepsilon$. Thus by the Darboux integrability criterion, we see that f is integrable.

Suppose that $I \neq \int_a^b f(x)\,dx$; without loss of generality $I < \int_a^b f(x)\,dx$. Since $U(P_n, f) \to I$, there exists $N \in \mathbb{N}$ such that $U(P_N, f) < \int_a^b f(x)\,dx$. This, however, is a contradiction of the very definition of integrable. Thus $I = \int_a^b f(x)\,dx$. □

Definition 3.25. *Let* $[a,b] \subset \mathbb{R}$.

1. *A* ***pointed partition*** $\widehat{P}$ *of interval* $[a,b]$ *consists of a partition* $P = \{x_i\}_{i=0}^n$ *of* $[a,b]$ *together with a set of points* $C = \{c_i\}_{i=1}^n$ *such that* $x_{i-1} \leq c_i \leq x_i$ *for each* $i = 1, \dots, n$.

2. *Suppose* $\widehat{P} = (P, C)$ *is a pointed partition of* $[a,b]$. *The* ***diameter*** *of* $\widehat{P}$, *which we denote* $\|\widehat{P}\|$, *is defined as the diameter of the partition* P.

3. *Suppose* $\widehat{P}$ *is a pointed partition of* $[a,b]$. *The* ***Riemann sum*** *of* f *with respect to* $\widehat{P}$, *which we denote* $R(\widehat{P}, f)$, *is the sum*

$$R(\widehat{P}, f) = \sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1})\,.$$

The collection of all pointed partitions of $[a,b]$ *is denoted* $\widehat{\mathscr{P}}([a,b])$.

Lemma 3.26. *Suppose* $\widehat{P} = (P, C)$ *is a pointed partition of* $[a,b]$ *and* $f \colon [a,b] \to \mathbb{R}$ *is a bounded function. Then*

$$L(P, f) \leq R(\widehat{P}, f) \leq U(P, f).$$

Proof. This follows directly from the simple observation that for each subinterval $[x_{i-1}, x_i]$ of $[a,b]$ and each $c_i \in [x_{i-1}, x_i]$ we have

$$\inf_{[x_{i-1},x_i]} f \leq f(c_i) \leq \sup_{[x_{i-1},x_i]} f.$$ □

Theorem 3.27 (Riemann convergence criterion)**.** *Suppose* $f \colon [a,b] \to \mathbb{R}$ *and* $I \in \mathbb{R}$. *Then the following are equivalent.*

1. f *is integrable and* $\int_a^b f(x)\,dx = I$.

2. *Whenever* $\widehat{P}_k$ *is a sequence of pointed partitions of* $[a,b]$ *such that* $\|\widehat{P}_k\| \to 0$ *then* $R(\widehat{P}_k, f) \to I$.

Proof. Suppose the first statement holds and let $\widehat{P}_k = (P_k, C_k)$ be a sequence of pointed partitions such that $\|\widehat{P}_k\| \to 0$. Thus $\|P_k\| \to 0$. By the Darboux convergence criterion we have $L(P_k, f) \to I$ and $U(P_k, f) \to I$. Since for each $k \in \mathbb{N}$ we have

$$L(P_k, f) \leq R(\widehat{P}_k, f) \leq U(P_k, f),$$

we immediately find that $R(\widehat{P}_k, f) \to I$. Thus the first statement implies the second.

Suppose now that the second statement holds and let P_k be a sequence of partitions such that $\|P_k\| \to 0$. Define pointed partitions $\widehat{P}_k^+ = (P_k, \overline{C}_k)$ and $\widehat{P}_k^- = (P_k, \underline{C}_k)$ as follows: For given partition $P_k = \{x_j^k\}_{j=0}^{n_k}$ let $\overline{c}_j^k$ be a point in $[x_{j-1}^k, x_j^k]$ such that

$$0 \le \left(\sup_{[x_{j-1}^k, x_j^k]} f \right) - f(\overline{c}_j^k) < \frac{1}{k \cdot n_k \|P_k\|}$$

and let $\underline{c}_j^k$ be a point in $[x_{j-1}^k, x_j^k]$ such that

$$0 \le f(\underline{c}_j^k) - \left(\inf_{[x_{j-1}^k, x_j^k]} f \right) < \frac{1}{k \cdot n_k \|P_k\|}.$$

Thus for each k we have

$$R(\widehat{P}_k^-) - \frac{1}{k} \le L(P_k, f) \le U(P_k, f) \le R(\widehat{P}_k^+, f) + \frac{1}{k}.$$

By hypothesis, we have $R(\widehat{P}_k^-) - \frac{1}{k} \to I$ and $R(\widehat{P}_k^+) + \frac{1}{k} \to I$. Thus the Darboux convergence criterion implies that the first statement holds. □

Remark 3.28. *The limit*

$$\lim_{\|\widehat{P}_k\| \to 0} R(\widehat{P}_k, f)$$

is called the Riemann integral of f.

Definition 3.29. *Suppose $f\colon [a,b] \to \mathbb{R}$ is integrable. We define*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proposition 3.30 (Properties of integrals)**.**

1. *(Linearity of integration I) If f, g are integrable on $[a,b]$, then so is $f+g$ and*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *(Linearity of integration II) If f is integrable on $[a,b]$ and $c \in \mathbb{R}$, then cf is Darboux integrable and*

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$$

3. *(Monotonicity of integration) If f, g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for each $x \in [a,b]$ then*

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

4. *(Additivity of intervals) Function f is integrable on intervals $[a,b]$ and $[b,c]$ if and only if it is Darboux integrable on $[a,c]$, in which case*

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx.$$

Proof. For the proof we make use of the Riemann convergence criterion. Let $\widehat{P}$ be a pointed partition of $[a,b]$. Observe the following properties of the Riemann sum:

1. If f and g are integrable on $[a,b]$ then

$$R(\widehat{P}, f+g) = R(\widehat{P}, f) + R(\widehat{P}, g).$$

2. If f is integrable on $[a,b]$ and $c \in \mathbb{R}$ then

$$R(\widehat{P}, cf) = cR(\widehat{P}, f).$$

3. If f and g are integrable with $f \leq g$ then

$$R(\widehat{P}, f) \leq R(\widehat{P}, g).$$

Applying these properties to each of a sequence of pointed partitions easily yields the first three properties.

To see the fourth property, we simply note that any partition of $[a,c]$ may be refined such that b is one of the points in the refinement. This leads us to conclude that for any partition P of $[a,c]$ we have

$$L(P,f) \leq \int_a^b f(x)\, dx + \int_b^c f(x)\, dx \leq U(P,f).$$

The desired results immediately follows.

$\square$

Lemma 3.31. *Suppose* $f\colon [a,b]\to\mathbb{R}$ *is integrable, and* $g\colon\mathbb{R}\to\mathbb{R}$ *is continuous. Then* $g\circ f\colon [a,b]\to\mathbb{R}$ *is integrable.*

Proof. Fix $\varepsilon > 0$. Since f is integrable, it is bounded; let $M\in\mathbb{R}$ such that $|f(x)|\le M$ for all $x\in[a,b]$. Since g is continuous and $[-M,M]$ is compact, we have $g\colon[-M,M]\to\mathbb{R}$ is uniformly continuous. Let $G=\max\{g(x)\mid |x|\le M\}$.

Let $\delta>0$ be such that $|x-y|<\delta$ implies

$$|g(x)-g(y)|<\frac{\varepsilon}{2(b-a)}.$$

Without loss of generality we may assume that $\delta\le\varepsilon/4G$.

By the Darboux integrability criterion, there exists a partion $P=\{x_j\}_{j=0}^n$ of $[a,b]$ such that $U(P,f)-L(P,f)<\delta^2$. For $1\le j\le n$

$$\overline{f}_j=\sup_{[x_{j-1},x_j]} f \quad\text{and}\quad \underline{f}_j=\inf_{[x_{j-1},x_j]} f.$$

Define subsets $J_{\text{small}}, J_{\text{big}}\subset\{1,\dots,n\}$ as follows: Let $J_{\text{small}}=\{j\mid \overline{f}_j-\underline{f}_j<\delta\}$ and $J_{\text{big}}=\{j\mid \overline{f}_j-\underline{f}_j\ge\delta\}$. Thus

$$\begin{aligned}&U(P,f)-L(P,f)\\&\qquad=\sum_{j\in J_{\text{small}}}(\overline{f}_j-\underline{f}_j)(x_j-x_{j-1})+\sum_{j\in J_{\text{big}}}(\overline{f}_j-\underline{f}_j)(x_j-x_{j-1})<\delta^2.\end{aligned}$$

Therefore

$$\delta\sum_{j\in J_{\text{big}}}(x_j-x_{j-1})\le\sum_{j\in J_{\text{big}}}(\overline{f}_j-\underline{f}_j)(x_j-x_{j-1})<\delta^2.$$

Consequently, using $\delta\le\varepsilon/4G$, we have

$$\begin{aligned}&\sum_{j\in J_{\text{big}}}\left(\sup_{[x_{j-1},x_j]}(g\circ f)-\inf_{[x_{j-1},x_j]}(g\circ f)\right)(x_j-x_{j-1})\\&\qquad\qquad\le 2G\sum_{j\in J_{\text{big}}}(x_j-x_{j-1})<2G\delta\le\frac{\varepsilon}{2}.\end{aligned}$$

Notice also that

$$\sum_{j\in J_{\text{small}}} \left(\sup_{[x_{j-1},x_j]} (g\circ f) - \inf_{[x_{j-1},x_j]} (g\circ f) \right) (x_j - x_{j-1})$$
$$\leq \sum_{j\in J_{\text{small}}} \left(\sup_{[\underline{f}_j,\overline{f}_k]} g - \inf_{[\underline{f}_j,\overline{f}_k]} g \right) (x_j - x_{j-1})$$
$$\leq \frac{\varepsilon}{2(b-a)} \sum_{j\in J_{\text{small}}} (x_j - x_{j-1}) < \frac{\varepsilon}{2}.$$

Thus

$$U(P, g\circ f) - L(P, g\circ f) < \varepsilon,$$

which implies that $g\circ f$ is integrable. □

Corollary 3.32. *If f is integrable on $[a,b]$ then so is $|f|$, with*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

Proof. That $|f|$ is integrable follows from the continuity of the absolute value function. The estimate follows from the fact that

$$-|f| \leq f \leq |f|$$

and the monotonicity of integration. □

Proposition 3.33 (Mean value theorem for integrals)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $c \in [a,b]$ such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof. Since f is continuous the extreme value theorem implies that there exists $x_{\min}, x_{\max} \in [a,b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

for all $x \in [a,b]$. The monotonicity of integration implies that

$$f(x_{\min})(b-a) \leq \int_a^b f(x)\,dx \leq f(x_{\max})(b-a).$$

Thus by the intermediate value theorem, there must exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$ □

Theorem 3.34 (Fundamental theorem of calculus)**.** *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is integrable on any closed interval contained in* $[a,b]$*.*

1. *The function* $F\colon [a,b] \to \mathbb{R}$ *defined by*

$$F(x) = \int_a^x f(t)\,dt$$

   is continuous. Furthermore, if f *is continuous then* F *is differentiable with* $F' = f$*.*

2. *Suppose* $F\colon [a,b] \to \mathbb{R}$ *is differentiable on* (a,b) *with* $F' = f$*. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof.

1. Let $F\colon [a,b] \to \mathbb{R}$ be defined as above. Since f must be bounded to be integrable, there exists $M \in \mathbb{R}$ such that $|f(x)| \le M$ for all $x \in [a,b]$. For each $x, y \in [a,b]$ with $x \le y$ we have

$$|F(x) - F(y)| = |\int_y^x f(t)\,dt| \le \int_y^x f(t)\,dt \le M|x-y|.$$

   This immediately implies the continuity of F.

   Suppose, now, that f is continuous; fix $x_* \in (a,b)$. For each $x \in (a,b)$ we have

$$F(x) - F(x_*) = \int_a^x f(t)\,dt - \int_a^{x_*} f(t)\,dt.$$

   If $x \ge x_*$ then

$$F(x) - F(x_*) = \int_{x_*}^x f(t)\,dt,$$

   while if $x < x_*$ we have

$$F(x) - F(x_*) = -\int_x^{x_*} f(t)\,dt.$$

   In either case, the mean value theorem for integrals implies that there exists c between x_* and x such that

$$F(x) - F(x_*) = f(c)(x - x_*).$$

If we take a sequence $x_k \to x_*$ then we obtain a sequence of numbers c_k between x_k and x_* such that

$$F(x_k) - F(x_*) = f(c_k)(x_k - x_*).$$

Since $x_k \to x_*$ we must have $c_k \to x_*$ as well. The continuity of f implies that $f(c_k) \to f(x_*)$. Thus we have

$$\frac{F(x_k) - F(x_*)}{x_k - x_*} \to f(x_*),$$

which implies that $F'(x_*) = f(x_*)$.

2. Let

$$g(x) = F(x) - \int_a^x f(t)\,dt.$$

The first part of the theorem implies that $g\colon [a,b] \to \mathbb{R}$ is continuous, and that g is differentiable on (a,b) with $g' = 0$. Thus g is a constant function. Since $g(a) = F(a)$ we conclude that

$$F(x) - \int_a^x f(t)\,dt = F(a)$$

for all $x \in [a,b]$. Evaluating this at $x = b$ yields the desired identity. □

Proposition 3.35 (Integration by parts)**.** *Suppose* $f, g\colon [a,b] \to \mathbb{R}$ *are continuous functions that are continuously differentiable on* (a,b)*. Then*

$$\int_a^b f(x)\,g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx.$$

Proof. This follows immediately from the product rule and the fundamental theorem of calculus. □

Proposition 3.36 (Change of variables)**.** *Suppose* $g\colon [a,b] \to \mathbb{R}$ *is continuous, is differentiable on* (a,b)*, and that* g' *extends to a continuous function on* $[a,b]$*. Suppose also that* f *is continuous on* $g([a,b])$*. Then*

$$\int_a^b (f \circ g)(x)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Proof. Let

$$F(x) = \int_a^x (f \circ g)(t)\, g'(t)\, dt - \int_{g(a)}^{g(x)} f(t)\, dt.$$

The fundamental theorem of calculus, and the fact that the composition of continuous functions is continuous, implies that $F\colon [a,b] \to \mathbb{R}$ is continuous. Furthermore, F is differentiable on (a,b) with $F' = 0$. Thus F is a constant function. Since $F(a) = 0$, we conclude that $F = 0$. Evaluating F at $x = b$ yields the desired identity. □

Proposition 3.37. *Suppose that for each $n = 1, 2, 3, \ldots$ we have an integrable function $f_n\colon [a,b] \to \mathbb{R}$ and suppose that f_n converges to $f\colon [a,b] \to \mathbb{R}$ with respect to the $L^\infty([a,b])$ norm. Then f is integrable and*

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

Proof. Let $\varepsilon > 0$. By $f_n \to f$ in $L^\infty([a,b])$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$f_n(x) - \frac{\varepsilon}{3(b-a)} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{3(b-a)} \qquad (\star)$$

for all $x \in [a,b]$.

Let P be a partition of $[a,b]$ such that

$$U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{3}.$$

We have

$$L(f_N, P) - \frac{\varepsilon}{3} \leq L(f, P) \leq U(f, P) \leq U(f_N, P) + \frac{\varepsilon}{3}.$$

Thus

$$U(f, P) - L(f, P) \leq U(f_N, P) - L(f_N, P) + \frac{2\varepsilon}{3} < \varepsilon.$$

From this we see that f is integrable.

To see that the integrals converge, note that applying the monotonicity of integration to $(\star)$ yields

$$\int_a^b f_n(x)\, dx - \frac{\varepsilon}{3} \leq \int_a^b f(x)\, dx \leq \int_a^b f_n(x) + \frac{\varepsilon}{3}$$

Thus for $n \geq N$ we have

$$\left| \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \right| < \varepsilon.$$

□