# Chapter 2

# Functions

## 2.1 Continuous functions

**Definition 2.1.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces. A function* $f\colon X \to Y$ *is* ***continuous at*** $\boldsymbol{x_* \in X}$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the property that for any* $x \in X$ *with* $d_X(x, x_*) < \delta$ *we have* $d(f(x), f(x_*)) < \varepsilon$.

*We say that* $\boldsymbol{f\colon X \to Y}$ ***is continuous*** *if* $f$ *is continuous at each* $x_* \in X$.

**Proposition 2.2** (Sequential and topological criteria for continuity)**.** *Suppose* $f\colon X \to Y$ *is a function between metric spaces. The following are equivalent:*

1. $f\colon X \to Y$ *is continuous.*

2. *(sequential criterion) For all sequences* $\{x_n\}_{n=1}^{\infty} \subset X$, *we have* $x_n \to x$ *in* $X$ *implies* $f(x_n) \to f(x)$ *in* $Y$.

3. *(topological criterion) For each open set* $V \subset Y$, *its preimage* $f^{-1}(V)$ *is open in* $X$.

4. *(alternate topological criterion) For each closed set* $V \subset Y$, *its preimage* $f^{-1}(V)$ *is closed in* $X$.

**Lemma 2.3.** *The composition of continuous functions is continuous.*

**Proposition 2.4.** *Suppose* $X, Y$ *are metric spaces,* $f\colon X \to Y$ *is a continuous function.*

1. *If* $U \subset X$ *is connected, then* $f(U)$ *is connected.*

*2. If $K \subset X$ is compact, then $f(K)$ is compact.*

**Definition 2.5.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f\colon X \to Y$ is* ***uniformly continuous*** *if for each $\varepsilon > 0$ there exists $\delta > 0$ with the property that for any $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ we have $d(f(x_1), f(x_2)) < \varepsilon$.*

**Proposition 2.6.** *Suppose $K, Y$ are metric spaces, $f\colon K \to Y$ is a continuous function, and $K$ is compact. Then $f$ is uniformly continuous.*

**Lemma 2.7.** *Suppose $\mathbb{V}$ is a normed vector space. The function $f\colon \mathbb{V} \to \mathbb{R}$ given by $f(x) = \|x\|$ is continuous.*

**Theorem 2.8** (Extreme value theorem)**.** *Let $K$ be a compact metric space and $\mathbb{V}$ be a normed vector space. Suppose that $f\colon K \to \mathbb{V}$ is continuous. Then there exists $x_{max} \in K$ such that*

$$\|f(x_{\max})\| = \sup\{\|f(x)\| \mid x \in K\}.$$

**Exercise 2.1.** Give an example of a continuous function $f\colon X \to Y$ and open set $U \subset X$ such that $f(U)$ is not open in $Y$.

**Exercise 2.2.** Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a connected set $Y \subset \mathbb{R}$ such that $f^{-1}(Y)$ is not connected.

**Exercise 2.3.** Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a compact set $Y \subset \mathbb{R}$ such that $f^{-1}(Y)$ is not compact.

**Exercise 2.4.** Give an example of a continuous function $f\colon X \to Y$ where $X$ is not connected, but $Y$ is connected.

**Exercise 2.5.** Give an example of a continuous, bijective function $f\colon X \to Y$ where $f^{-1}\colon Y \to X$ is not continuous.

**Exercise 2.6.** Let $d(\cdot, \cdot)$ be the standard metric on $\mathbb{R}$ and let $Y = \{x \in \mathbb{R} \mid x > 1\}$. Draw a plot of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = d(x, Y)$.

**Exercise 2.7.** Show that the function $f\colon (0, \infty) \to (0, \infty)$ given by $f(x) = 2x$ is uniformly continuous.

**Exercise 2.8.** Show that the function $f\colon (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

**Exercise 2.9.** Show that the function $f\colon (2, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$ is uniformly continuous.

**Exercise 2.10.** Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. Show that there exists $x_{\max}, x_{\min} \in [a, b]$ such that

$$f(x_{\max}) = \sup\{f(x) \mid x \in [a, b]\} \quad \text{and} \quad f(x_{\min}) = \inf\{f(x) \mid x \in [a, b]\}.$$

## 2.2 Path connectedness

**Definition 2.9.** *A metric space $X$ is called* ***path connected*** *if for each $x_0, x_1 \in X$ there exists a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.*

**Proposition 2.10.** *Suppose $f\colon X \to Y$ is continuous and $U \subset X$ is path connected. Then $f(U)$ is path connected.*

**Proposition 2.11.** *If metric space $X$ is path connected then it is connected.*

**Proposition 2.12.** *Not every connected metric space is path connected*

## 2.3 Homeomorphisms

**Lemma 2.13.** *Continuity of a function is a topological definition.*

**Definition 2.14.** *Topological spaces $X$ and $Y$ are called* ***homeomorphic*** *if there exists a continuous bijection $X \to Y$ such that its inverse is also continuous. Such a bijection is called a* ***homeomorphism****. If $X$ and $Y$ are homeomorphic we write $X \cong Y$.*

**Lemma 2.15.** *Being homeomorphic is an equivalence relation on the set of topological spaces.*

**Definition 2.16.** *A property of topological spaces is called* ***homeomorphism invariant*** *if whenever $X$ has that property and $X \cong Y$ then $Y$ also has that property.*

**Proposition 2.17.**

1. *The property of being compact is homeomorphism invariant.*
2. *The property of being connected is homeomorphism invariant.*
3. *The property of being path connected is homeomorphism invariant.*

**Exercise 2.11.**

1. Prove that $(0,1)$ is homeomorphic to $\mathbb{R}$.
2. Prove that $[0,1]$ is not homeomorphic to $\mathbb{R}$.

## 2.4 Contraction mappings

**Definition 2.18.** *Suppose $X$ is a metric space with metric $d$. A function $f\colon X \to X$ is called a* ***contraction mapping*** *if there exists constant $\alpha \in [0,1)$ such that $d(f(x), f(y)) \le \alpha\, d(x,y)$ for all $x, y \in X$.*

**Lemma 2.19.** *Let $X$ be a metric space and suppose $f\colon X \to X$ is a contraction mapping. Then $f$ is uniformly continuous.*

**Theorem 2.20** (Banach fixed point theorem)**.** *Suppose $X$ is a complete, nonempty metric space. If $f\colon X \to X$ is a contraction mapping, then there exists a unique $x_* \in X$ such that $f(x_*) = x_*$.*

**Exercise 2.12.** Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that has a fixed point, but that is not a contraction mapping.

## 2.5 Spaces of continuous functions

**Definition 2.21.** *Suppose $X$ is a metric space and $\mathbb{V}$ is a normed vector space.*

1. *We denote the* ***set of bounded functions*** *$X \to \mathbb{R}$ by $L^\infty(X;\mathbb{V})$.*[1]
2. *We denote the* ***set of continuous functions*** *$X \to \mathbb{R}$ by $C^0(X;\mathbb{V})$.*
3. *We denote the* ***set of bounded, continuous functions*** *$X \to \mathbb{R}$ by $C^0_b(X;\mathbb{V})$; thus $C^0_b(X;\mathbb{V}) = L^\infty(X;\mathbb{V}) \cap C^0(X;\mathbb{V})$.*

*If $\mathbb{V} = \mathbb{R}$, we simply write $L^\infty(X)$, $C^0(X)$, $C^0_b(X)$ for these sets.*

**Lemma 2.22.** *Let $\mathbb{V}$ be a normed vector space. If $K$ is a compact metric space then $C^0_b(K;\mathbb{V}) = C^0(K;\mathbb{V})$.*

**Definition 2.23.** *Suppose $X$ is a metric space and $\mathbb{V}$ is a normed vector space.*

1. *For any function $f\colon X \to \mathbb{V}$ we define the* ***supremum norm of f***[2] *to be*

$$\|f\|_{L^\infty(X;\mathbb{V})} = \sup\{\|f(x)\| \mid x \in X\}.$$

[1] We are abusing the standard notation here. Technically, the symbol $L^\infty(X;\mathbb{V})$ should be reserved for the (larger) set of *essentially bounded* functions. However, the definition of this set requires measure theory, which is beyond the scope of these notes.

[2] Again, we are abusing the standard notation. The $L^\infty$ norm should be defined using the essential supremum, a topic beyond the scope of these notes.

2. *We say that a sequence of functions $f_n\colon X \to \mathbb{V}$* ***converges uniformly*** *to function $f\colon X \to \mathbb{V}$ if $\|f_n - f\|_{L^\infty(X;\mathbb{V})} \to 0$ as a sequence of real numbers.*

*Uniform convergence is also called* ***convergence in $L^\infty$****.*

**Lemma 2.24.** *Let $X$ be a metric space and $\mathbb{V}$ be a normed vector space. The supremum norm makes each of the sets $L^\infty(X;\mathbb{V})$, $C^0(X;\mathbb{V})$, $C^0_b(X;\mathbb{V})$ normed vector spaces, and thus metric spaces with respect to the metric*

$$d(f,g) = \|f-g\|_{L^\infty(X;\mathbb{V})}.$$

**Proposition 2.25.** *Let $X$ be a complete metric space and $\mathbb{V}$ be a Banach space. Then $C^0_b(X;\mathbb{V})$ is a Banach space.*

## 2.6 Real valued functions

**Theorem 2.26** (Intermediate value theorem)**.** *Suppose $X$ is a connected metric space and $f\colon X \to \mathbb{R}$ is continuous. Let $a, b \in X$. For any number $y$ between $f(a)$ and $f(b)$ there exists $x \in X$ such that $f(x) = y$.*

**Corollary 2.27.** *Suppose $f\colon [a,b] \to [a,b]$ is continuous. Then there exists $x \in [a,b]$ such that $f(x) = x$.*

**Definition 2.28.** *Let $X$ be a metric space.*

1. *We say that a sequence $\{f_k\}_{k=1}^\infty \subset C^0(X)$ is* ***pointwise bounded*** *if for each $x \in X$ there exists $M$ such that $|f_k(x)| \le M$ for all $k \in \mathbb{N}$.*

2. *We say that a sequence $\{f_k\}_{k=1}^\infty \subset C^0(X)$ is* ***uniformly bounded*** *if there exists $M$ such that $\|f_k\|_{L^\infty(X)} \le M$ for all $k \in \mathbb{N}$.*

3. *We say that a sequence $\{f_k\}_{k=1}^\infty \subset C^0(X)$ is* ***equicontinuous*** *if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ we have*

$$d(x,y) < \delta \quad \Longrightarrow \quad |f_k(x) - f_k(y)| < \varepsilon.$$

**Proposition 2.29.** *Suppose $K \subset \mathbb{R}$ is compact and $\{f_n\}_{n=1}^\infty \subset C^0(K)$ is a convergent sequence. Then $\{f_n\}_{n=1}^\infty$ is equicontinuous.*

**Lemma 2.30.** *Suppose $K$ is a compact metric space. If $\{f_k\}_{k=1}^\infty \subset C^0(K)$ is equicontinuous and pointwise bounded then the sequence is uniformly bounded.*

**Theorem 2.31** (Arzela-Ascoli)**.** *Suppose $K \subset \mathbb{R}$ is compact and $\{f_k\}_{k=1}^{\infty} \subset C^0(K)$ is both uniformly bounded and equicontinuous. Then $\{f_k\}_{k=1}^{\infty}$ is bounded in $C^0(K)$ and contains a subsequence that converges to some function $f \in C^0(K)$.*

**Remark 2.32.** *The Arzela-Ascoli theorem holds when $K$ is any compact metric space. The proof relies on showing that any compact metric space has a countable dense subset. See, for example, the book* Principles of Mathematical Analysis *by Walter Rudin.*

**Exercise 2.13.** Find a subset $U \subset \mathbb{R}$ and a sequence of functions $f_n : U \to \mathbb{R}$ that is pointwise bounded but not uniformly bounded.