Chapter 2

Functions

2.1 Continuous functions

Definition 2.1. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A function* $f\colon X \to Y$ *is* ***continuous at*** $\boldsymbol{x_* \in X}$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the property that for any* $x \in X$ *with* $d_X(x, x_*) < \delta$ *we have* $d(f(x), f(x_*)) < \varepsilon$.

We say that $\boldsymbol{f\colon X \to Y}$ ***is continuous*** *if* f *is continuous at each* $x_* \in X$.

Proposition 2.2 (Sequential and topological criteria for continuity)**.** *Suppose* $f\colon X \to Y$ *is a function between metric spaces. The following are equivalent:*

1. $f\colon X \to Y$ *is continuous.*

2. *(sequential criterion) For all sequences* $\{x_n\}_{n=1}^{\infty} \subset X$*, we have* $x_n \to x$ *in* X *implies* $f(x_n) \to f(x)$ *in* Y.

3. *(topological criterion) For each open set* $V \subset Y$*, its preimage* $f^{-1}(V)$ *is open in* X.

4. *(alternate topological criterion) For each closed set* $V \subset Y$*, its preimage* $f^{-1}(V)$ *is closed in* X.

Proof. To see that the first statement implies the second, fix $\varepsilon > 0$ and let $\delta > 0$ be as in the definition of continuity. Suppose that $x_n \to x$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d_X(x_n, x) < \delta$. For such n, the continuity of f implies that $d_Y(f(x_n), f(x)) < \varepsilon$. Hence $f(x_n) \to f(x)$ in Y.

We now show that the sequential criterion implies the alternate topological criterion. Let $V \subset Y$ be a closed set and suppose that $\{x_n\}_{n=1}^{\infty} \subset f^{-1}(V)$ is a sequence such that $x_n \to x \in X$. Thus $\{f(x_n)\}_{n=1}^{\infty} \subset V$ is a sequence that, by hypothesis, converges to $f(x)$. Since V is closed, this implies $f(x) \in V$ and thus $x \in f^{-1}(V)$. Consequently $f^{-1}(V)$ is closed.

Since $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$, the alternate topological criterion for continuity is clearly equivalent to the topological criterion.

Finally, we show that the topological criterion for continuity implies the definition. Supposing the topological criterion holds, fix $x_* \in X$ and let $\varepsilon > 0$. By hypothesis $f^{-1}(B_\varepsilon(f(x_*)))$ is open in X and contains x. Thus there exists $\delta > 0$ such that $B_\delta(x_*) \subset f^{-1}(B_\varepsilon(f(x_*)))$. This immediately implies that if $d_X(x, x_*) < \delta$ then $d_Y(f(x), f(x_*)) < \varepsilon$. □

Lemma 2.3. *The composition of continuous functions is continuous.*

Proof. This follows immediately from the topological criterion for continuity. □

Proposition 2.4. *Suppose X, Y are metric spaces, $f\colon X \to Y$ is a continuous function.*

1. *If $U \subset X$ is connected, then $f(U)$ is connected.*
2. *If $K \subset X$ is compact, then $f(K)$ is compact.*

Proof.

1. Suppose $f(U) = V_1 \sqcup V_2$, where $V_1, V_2 \subset Y$ are open. Then $U = U_1 \sqcup U_2$, where $U_i = U \cap f^{-1}(V_i)$ are open sets. The connectedness of U implies that either U_1 or U_2 is empty; without loss of generality $U_1 = \emptyset$. This implies that $V_1 = \emptyset$ and thus that $f(U)$ is connected.
2. Let $\{U_j\}_{j \in J}$ be an open cover of $f(K)$. Then $\{f^{-1}(U_j)\}_{j \in J}$ is an open cover of K. The compactness of K implies that there exists a finite subset $L \subset J$ such that $K \subset \bigcup_{j \in L} f^{-1}(U_j)$. Thus $f(K) \subset \bigcup_{j \in L} U_j$, which implies that $\{U_j\}$ has a finite subcover. Hence $f(K)$ is compact. □

Definition 2.5. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is* ***uniformly continuous*** *if for each $\varepsilon > 0$ there exists $\delta > 0$ with the property that for any $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ we have $d(f(x_1), f(x_2)) < \varepsilon$.*

Proposition 2.6. *Suppose K, Y are metric spaces, $f\colon K \to Y$ is a continuous function, and K is compact. Then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Since $f\colon K \to Y$ is continuous, then for each $x \in K$ there exists $\delta(x) > 0$ such that $x' \in B_{\delta(x)}(x)$ implies that $f(x') \in B_{\varepsilon/2}(f(x))$. Notice that $\{B_{\delta(x)}(x) \mid x \in K\}$ is an open cover of K. Thus there exists a finite subcover $B_{\delta(x_1)}(x_1), \dots, B_{\delta(x_N)}(x_N)$. Let δ be the Lebesgue number for this subcover. Thus if $d(x, x') < \delta$ then there exists $k \in \{1, \dots, N\}$ such that $x, x' \in B_{\delta(x_k)}(x_k)$, which implies that $f(x), f(x') \in B_{\varepsilon/2}(f(x_k))$. The triangle inequality implies that $d(f(x), f(x')) < \varepsilon$. □

Lemma 2.7. *Suppose $\mathbb{V}$ is a normed vector space. The function $f\colon \mathbb{V} \to \mathbb{R}$ given by $f(x) = \|x\|$ is continuous.*

Proof. This follows directly from the definition of the metric arising from the norm on $\mathbb{V}$. □

Theorem 2.8 (Extreme value theorem)**.** *Let K be a compact metric space and $\mathbb{V}$ be a normed vector space. Suppose that $f\colon K \to \mathbb{V}$ is continuous. Then there exists $x_{max} \in K$ such that*

$$\|f(x_{\max})\| = \sup\{\|f(x)\| \mid x \in K\}.$$

Proof. Let $f_{\max} = \sup\{\|f(x)\| \mid x \in K\}$. Then there exists a sequence $\{x_n\}_{k=1}^{\infty} \subset K$ such that $\|f(x_n)\| \to f_{\max}$. By the sequential criterion for compactness, there exists a subsequence $x_{n_k} \to x_{\min} \in K$. The continuity of $x \mapsto \|f(x)\|$ implies that $\|f(x_{n_k})\| \to \|f(x)\|$, and thus $\|f(x)\| = f_{\max}$. □

Exercise 2.1. Give an example of a continuous function $f\colon X \to Y$ and open set $U \subset X$ such that $f(U)$ is not open in Y.

Exercise 2.2. Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a connected set $Y \subset \mathbb{R}$ such that $f^{-1}(Y)$ is not connected.

Exercise 2.3. Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a compact set $Y \subset \mathbb{R}$ such that $f^{-1}(Y)$ is not compact.

Exercise 2.4. Give an example of a continuous function $f\colon X \to Y$ where X is not connected, but Y is connected.

Exercise 2.5. Give an example of a continuous, bijective function $f\colon X \to Y$ where $f^{-1}\colon Y \to X$ is not continuous.

Exercise 2.6. Let $d(\cdot,\cdot)$ be the standard metric on $\mathbb{R}$ and let $Y = \{x \in \mathbb{R} \mid x > 1\}$. Draw a plot of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = d(x, Y)$.

Exercise 2.7. Show that the function $f\colon (0,\infty) \to (0,\infty)$ given by $f(x) = 2x$ is uniformly continuous.

Exercise 2.8. Show that the function $f\colon (0,\infty) \to (0,\infty)$ given by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Exercise 2.9. Show that the function $f\colon (2,\infty) \to (0,\infty)$ given by $f(x) = \frac{1}{x}$ is uniformly continuous.

Exercise 2.10. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Show that there exists $x_{\max}, x_{\min} \in [a,b]$ such that

$$f(x_{\max}) = \sup\{f(x) \mid x \in [a,b]\} \quad \text{and} \quad f(x_{\min}) = \inf\{f(x) \mid x \in [a,b]\}.$$

2.2 Path connectedness

Definition 2.9. *A metric space X is called* ***path connected*** *if for each $x_0, x_1 \in X$ there exists a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.*

Proposition 2.10. *Suppose $f\colon X \to Y$ is continuous and $U \subset X$ is path connected. Then $f(U)$ is path connected.*

Proof. Let $y_0, y_1 \in f(U)$ and choose $x_0 \in f^{-1}(y_0) \cap U$, $x_1 \in f^{-1}(y_1) \cap U$. The path connectedness of U implies that there exists $\gamma\colon [0,1] \to U$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The function $\Gamma = f \circ \gamma$ is a continuous function $[0,1] \to f(U)$ such that $\Gamma(0) = y_0$ and $\Gamma(1) = y_1$. Thus $f(U)$ is path connected. □

Proposition 2.11. *If metric space X is path connected then it is connected.*

Proof. We argue by contradiction and suppose that X is path connected, but not connected. Then $X = U_0 \sqcup U_1$, where $U_0, U_1 \subset X$ are open and nonempty. Let $x_0 \in U_0$ and $x_1 \in U_1$. By the path connectedness of U, there exists continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The continuity of γ implies that $V_0 = \gamma^{-1}(U_0)$ and $V_1 = \gamma^{-1}(U_1)$ are open. Notice also that $0 \in V_0$, $1 \in V_1$, and that $[0,1] = V_0 \sqcup V_1$. This, however, contradicts the connectedness of $[0,1]$. □

Proposition 2.12. *Not every connected metric space is path connected*

Proof. It suffices to provide an example. We present here an adaptation of the "flea and comb" example from the book *A first course in algebraic topology* by C. Kosniowski.

Let $F = \{(0,1)\} \subset \mathbb{R}^2$ be the "flea" and define the "comb" to be

$$C = \{(x,0) \in \mathbb{R}^2 \mid x \in [0,1]\} \bigcup_{n\in\mathbb{N}} \{(1/n, y) \in \mathbb{R}^2 \mid y \in [0,1]\}.$$

Let $X = F \cup C$, which we view as a subset of the metric space $\mathbb{R}^2$; we use the metric induced by the standard metric.

It is straightforward to see that C is a path connected, and hence connected, subspace of X. Suppose, however, that X is not connected and thus $X = U_0 \sqcup U_1$ for some open, nonempty sets U_0, U_1. Without loss of generality, $F \subset U_0$ and thus $U_1 \subset C$. By definition of U_0 being open there exists $r > 0$ such that

$$B_r(0,1) = \{(x,y) \in X \mid x^2 + (y-1)^2 < r^2\} \subset U_0.$$

Since $B_r(0,1)$ is open in X, we the set $V = B_r(0,1) \cap C$ is open in C. Notice also that V is not empty. Thus $C = V \sqcup U_1$, the disjoint union of two nonempty open sets, which contradicts the connectedness of C. From this we conclude that X is connected.

To see that X is not path connected, we argue by contradiction and suppose that there exists a continuous path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = (0,1) \in F$ and $\gamma(1) = (1,0) \in C$. Let $W = X \cap \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. Since $W \subset X$ is open and γ is continuous the set $\gamma^{-1}(W)$ is open in $[0,1]$. Thus for each $t_* \in \gamma^{-1}(0,1) \subset \gamma^{-1}(W)$ there exists $\delta > 0$ such that $B_\delta(t_*) \subset \gamma^{-1}(W)$.

We now claim that for such a t_* we have $\gamma(B_\delta(t_*)) \subset F$. Suppose that this is not the case. Then there exists $\hat{t} \in B_\delta(t_*)$ such that $\gamma(t) = (\hat{x}, \hat{y})$ with $\hat{x} > 0$. Let $z \in (0,1)$ such that $z < \hat{x}$. By intersecting with the sets $\{(x,y) \mid x > z\}$ and $\{(x,y) \mid x < z\}$, the set $\gamma(B_\delta(t_*))$ can be realized as the disjoint union of two nonempty open sets. This means that $\gamma(B_\delta(t_*))$ is not connected, which contradicts our theorem that the continuous image of a connected set is connected. Consequently, we see that for any $t_* \in \gamma^{-1}(0,1)$ there exists $\delta > 0$ such that $\gamma(B_\delta(t_*)) \subset F$.

The previous argument shows that $\gamma^{-1}(F)$ is open. However F is a closed subset of X and thus $\gamma^{-1}(F)$ is closed. But the interval $[0,1]$ is connected, which means that the only nonempty subset that is both open and closed in the whole set itself. This contradicts our assumption that $\gamma(1) \in C$. Thus we see that X cannot be path connected. $\square$

2.3 Homeomorphisms

Lemma 2.13. *Continuity of a function is a topological definition.*

Definition 2.14. *Topological spaces X and Y are called* ***homeomorphic*** *if there exists a continuous bijection $X \to Y$ such that its inverse is also continuous. Such a bijection is called a* ***homeomorphism****. If X and Y are homeomorphic we write $X \cong Y$.*

Lemma 2.15. *Being homeomorphic is an equivalence relation on the set of topological spaces.*

Proof. This follows immediately from the identity map being a homeomorphism and since compositions of continuous bijections are themselves continuous bijections. □

Definition 2.16. *A property of topological spaces is called* ***homeomorphism invariant*** *if whenever X has that property and $X \cong Y$ then Y also has that property.*

Proposition 2.17.

1. *The property of being compact is homeomorphism invariant.*
2. *The property of being connected is homeomorphism invariant.*
3. *The property of being path connected is homeomorphism invariant.*

Proof. The proposition follows from the facts that the continuous image of a compact set is compact, the continuous image of a connected set is connected, and the continuous image of a path connected set is path connected. □

Exercise 2.11.

1. Prove that $(0, 1)$ is homeomorphic to $\mathbb{R}$.
2. Prove that $[0, 1]$ is not homeomorphic to $\mathbb{R}$.

2.4 Contraction mappings

Definition 2.18. *Suppose X is a metric space with metric d. A function $f : X \to X$ is called a* ***contraction mapping*** *if there exists constant $\alpha \in [0, 1)$ such that $d(f(x), f(y)) \leq \alpha\, d(x, y)$ for all $x, y \in X$.*

Lemma 2.19. *Let X be a metric space and suppose $f\colon X \to X$ is a contraction mapping. Then f is uniformly continuous.*

Proof. This follows immediately from the definition of contraction mapping. □

Theorem 2.20 (Banach fixed point theorem)**.** *Suppose X is a complete, nonempty metric space. If $f\colon X \to X$ is a contraction mapping, then there exists a unique $x_* \in X$ such that $f(x_*) = x_*$.*

Proof. The proof proceeds by using the contraction property to inductively construct a Cauchy sequence converging to a fixed point.

Define the sequence $\{x_n\}_{n=1}^{\infty} \subset X$ by choosing x_1 arbitrarily and setting $x_{n+1} = f(x_n)$. The contraction property implies that for any $n \in \mathbb{N}$ we have

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1}) \leq \alpha d(x_n, x_{n+1}).$$

Thus by induction we see that for $m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=0}^{m-n-1} d(x_{n+k}, x_{n+k+1}) \\ &\leq d(x_n, x_{n+1}) \sum_{k=0}^{m-n-1} \alpha^k \\ &\leq \frac{d(x_1, x_2)}{1-\alpha}\, \alpha^{n-1}. \end{aligned}$$

Since the right side tends to zero as $n \to \infty$ we see that the sequence $\{x_n\}$ is Cauchy and thus converges to some $x \in X$.

Since f is continuous $x_n \to x$ implies that $f(x_n) \to f(x)$. Thus taking the limit of the defining relation $x_{n+1} = f(x_n)$ implies that $x = f(x)$.

Finally, we establish uniqueness. Suppose that $f(x) = x$ and $f(x') = x'$. Then $d(x, x') = d(f(x), f(x')) \leq \alpha d(x, x')$. This implies $(1-\alpha)d(x, x') \leq 0$; thus $x = x'$. □

Exercise 2.12. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that has a fixed point, but that is not a contraction mapping.

2.5 Spaces of continuous functions

Definition 2.21. *Suppose X is a metric space and $\mathbb{V}$ is a normed vector space.*

1. *We denote the* ***set of bounded functions*** $X \to \mathbb{R}$ *by* $L^\infty(X;\mathbb{V})$.[1]

2. *We denote the* ***set of continuous functions*** $X \to \mathbb{R}$ *by* $C^0(X;\mathbb{V})$.

3. *We denote the* ***set of bounded, continuous functions*** $X \to \mathbb{R}$ *by* $C^0_b(X;\mathbb{V})$*; thus* $C^0_b(X;\mathbb{V}) = L^\infty(X;\mathbb{V}) \cap C^0(X;\mathbb{V})$.

If $\mathbb{V} = \mathbb{R}$*, we simply write* $L^\infty(X)$*,* $C^0(X)$*,* $C^0_b(X)$ *for these sets.*

Lemma 2.22. *Let* $\mathbb{V}$ *be a normed vector space. If* K *is a compact metric space then* $C^0_b(K;\mathbb{V}) = C^0(K;\mathbb{V})$.

Proof. It suffices to show that each function $f \in C^0(K;\mathbb{V})$ is bounded. This follows immediately from applying the extreme value theorem to the function $K \to \mathbb{R}$ given by $x \mapsto \|f(x)\|$. □

Definition 2.23. *Suppose* X *is a metric space and* $\mathbb{V}$ *is a normed vector space.*

1. *For any function* $f\colon X \to \mathbb{V}$ *we define the* ***supremum norm of*** $\boldsymbol{f}$[2] *to be*
$$\|f\|_{L^\infty(X;\mathbb{V})} = \sup\{\|f(x)\| \mid x \in X\}.$$

2. *We say that a sequence of functions* $f_n\colon X \to \mathbb{V}$ ***converges uniformly*** *to function* $f\colon X \to \mathbb{V}$ *if* $\|f_n - f\|_{L^\infty(X;\mathbb{V})} \to 0$ *as a sequence of real numbers.*

Uniform convergence is also called ***convergence in*** $\boldsymbol{L^\infty}$.

Lemma 2.24. *Let* X *be a metric space and* $\mathbb{V}$ *be a normed vector space. The supremum norm makes each of the sets* $L^\infty(X;\mathbb{V})$*,* $C^0(X;\mathbb{V})$*,* $C^0_b(X;\mathbb{V})$ *normed vector spaces, and thus metric spaces with respect to the metric*
$$d(f,g) = \|f - g\|_{L^\infty(X;\mathbb{V})}.$$

Proof. Clearly all three sets are real vector spaces. The positive definite and scaling properties of a norm are evident, so we only need to address the triangle inequality. Let $f, g\colon X \to \mathbb{V}$. For each $x \in X$ we have
$$\|f(x) + g(x)\| \le \|f(x)\| + \|g(x)\|.$$

[1] We are abusing the standard notation here. Technically, the symbol $L^\infty(X;\mathbb{V})$ should be reserved for the (larger) set of *essentially bounded* functions. However, the definition of this set requires measure theory, which is beyond the scope of these notes.

[2] Again, we are abusing the standard notation. The L^∞ norm should be defined using the essential supremum, a topic beyond the scope of these notes.

This implies that

$$\|f(x) + g(x)\| \leq \|f\|_{L^\infty(X;\mathbb{V})} + \|g\|_{L^\infty(X;\mathbb{V})},$$

which implies the desired result. □

Proposition 2.25. *Let X be a complete metric space and $\mathbb{V}$ be a Banach space. Then $C_b^0(X;\mathbb{V})$ is a Banach space.*

Proof. Let $\{f_n\}_{n=1}^\infty \subset C_b^0(X;\mathbb{V})$ be a Cauchy sequence. We first claim that there exists $M \in \mathbb{R}$ such that $\|f(x)\| \leq M$ for all $x \in M$. By definition of Cauchy sequence, there exists $N \in \mathbb{N}_1$ such that $n, m \geq N_1$ implies $\|f_n - f_m\|_{L^\infty(X;\mathbb{V})} \leq 1$. From this it follows that for any $n \in \mathbb{N}$ we have

$$\|f_n(x)\| \leq 1 + \max\{\|f_n\|_{L^\infty(X;\mathbb{V})} \mid 1 \leq n \leq N_1\} = M.$$

Next, we construct a bounded function $f\colon X \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in X$. Fix $x \in X$. The fact that $\{f_n\}$ is Cauchy in $C_b^0(X;\mathbb{V})$ implies that the sequence $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{V}$. Define function $f\colon X \to \mathbb{V}$ by setting $f(x)$ to be the limit of the the sequence $\{f_n(x)\}$. Since $\|f_n(x)\| \leq M$ is bounded for all $n \in \mathbb{N}$ and for all $x \in X$, we have that $\|f(x)\| \leq M$ for all x. Thus the function f is bounded.

We now show that $f_n \to f$ in L^∞. Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $n, m \geq N$ implies $\|f_n - f_m\|_{L^\infty(X;\mathbb{V})} < \varepsilon/4$. Since for each $x \in X$ the sequence $\{f_N(x) - f_m(x)\}_{m=N}^\infty$ converges to $f_N(x) - f(x)$, we have $\|f_N(x) - f(x)\| < \varepsilon/4$ for all $x \in X$. Let $n \geq N$ and fix $x \in X$. Then for all $m \geq N$ we have

$$\|f_n(x) - f(x)\| \leq \|f_n(x) - f_m(x)\| + \|f_m(x) - f(x)\| < \frac{\varepsilon}{2} + \|f_m(x) - f(x)\|.$$

Notice that as $m \to \infty$ we have $\|f_m(x) - f(x)\| \to 0$. Thus for all $n \geq N$ we have

$$\|f_n(x) - f(x)\| \leq \frac{\varepsilon}{4}$$

for all $x \in X$. Taking the supremum over all $x \in X$, we find that $n \geq N$ implies $\|f_n - f\|_{L^\infty(X;\mathbb{V})} \leq \varepsilon/4 < \varepsilon$.

It remains to show that f is continuous. Let N be as above, and let $x_1, x_2 \in X$. We have

$$\begin{aligned}
&\|f(x_1) - f(x_2)\| \\
&\quad \leq \|f(x_1) - f_N(x_1)\| + \|f_N(x_1) - f_N(x_2)\| + \|f_N(x_2) - f(x_2)\| \\
&\qquad < \frac{\varepsilon}{2} + \|f_N(x_1) - f_N(x_2)\|.
\end{aligned}$$

The continuity of f_N implies that there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies that $\|f_N(x_1) - f_N(x_2)\| < \varepsilon/2$, which completes the proof. □

2.6 Real valued functions

Theorem 2.26 (Intermediate value theorem)**.** *Suppose X is a connected metric space and $f\colon X \to \mathbb{R}$ is continuous. Let $a, b \in X$. For any number y between $f(a)$ and $f(b)$ there exists $x \in X$ such that $f(x) = y$.*

Proof. The proof proceeds by contradiction. Suppose there exists y between $f(a)$ and $f(b)$ such that $f^{-1}(y) = \emptyset$. Then $X = f^{-1}(-\infty, y) \sqcup f^{-1}(y, \infty)$, with both sets being open and nonempty. This contradicts the connectedness of X. □

Corollary 2.27. *Suppose $f\colon [a, b] \to [a, b]$ is continuous. Then there exists $x \in [a, b]$ such that $f(x) = x$.*

Proof. If $f(a) = a$ or $f(b) = b$ then we are done. Otherwise, $f(a)-a > 0$ and $f(b) - b < 0$. The existence of $x \in [a, b]$ such that $f(x) - x = 0$ now follows from applying the intermediate value theorem to the function $g\colon [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. □

Definition 2.28. *Let X be a metric space.*

1. *We say that a sequence $\{f_k\}_{k=1}^{\infty} \subset C^0(X)$ is **pointwise bounded** if for each $x \in X$ there exists M such that $|f_k(x)| \leq M$ for all $k \in \mathbb{N}$.*
2. *We say that a sequence $\{f_k\}_{k=1}^{\infty} \subset C^0(X)$ is **uniformly bounded** if there exists M such that $\|f_k\|_{L^\infty(X)} \leq M$ for all $k \in \mathbb{N}$.*
3. *We say that a sequence $\{f_k\}_{k=1}^{\infty} \subset C^0(X)$ is **equicontinuous** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ we have*

$$d(x, y) < \delta \quad \Longrightarrow \quad |f_k(x) - f_k(y)| < \varepsilon.$$

Proposition 2.29. *Suppose $K \subset \mathbb{R}$ is compact and $\{f_n\}_{n=1}^{\infty} \subset C^0(K)$ is a convergent sequence. Then $\{f_n\}_{n=1}^{\infty}$ is equicontinuous.*

Proof. Let $\varepsilon > 0$. Since $\{f_n\}$ is a convergent sequence it is Cauchy and thus there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $\|f_n - f_m\|_{L^\infty(K)} < \varepsilon/3$.

Note that since K is compact, each function f_n is uniformly continuous. Thus for each $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that

$$d(x, y) < \delta_n \quad \text{implies} \quad |f_n(x) - f_n(y)| < \varepsilon/3.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_N\}$.

Suppose that $x, y \in K$ are such that $d(x, y) < \delta$. If $n \in \{1, \dots, N\}$ then by construction $|f_n(x) - f_n(y)| < \varepsilon$. If $n > N$, then the supremum norm estimate implies that

$$\begin{aligned}|f_n(x) - f_n(y)| \\ &\le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

which concludes the proof. □

Lemma 2.30. *Suppose K is a compact metric space. If $\{f_k\}_{k=1}^{\infty} \subset C^0(K)$ is equicontinuous and pointwise bounded then the sequence is uniformly bounded.*

Proof. Since the sequence is equicontinuous there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < 1$ for all $n \in \mathbb{N}$.

The compactness of K implies that there exist a finite number of points $x_1, \dots, x_N$ such that $\{B_\delta(x_i)\}_{i=1}^{N}$ is an open cover of K. For each x_i there exists M_i such that $|f_n(x_i)| \le M_i$ for all $n \in \mathbb{N}$.

Let $M = \max\{M_1, \dots, M_N\}$. The triangle inequality implies that $|f_n(x)| \le M + 1$ for all $x \in K$ and $n \in \mathbb{N}$. □

Theorem 2.31 (Arzela-Ascoli)**.** *Suppose $K \subset \mathbb{R}$ is compact and $\{f_k\}_{k=1}^{\infty} \subset C^0(K)$ is both uniformly bounded and equicontinuous. Then $\{f_k\}_{k=1}^{\infty}$ is bounded in $C^0(K)$ and contains a subsequence that converges to some function $f \in C^0(K)$.*

Proof. Since $\mathbb{Q}$ is countable and is dense in $\mathbb{R}$, we may choose a sequence $\{q_i\}_{i=1}^{\infty} \subset K$ such that $\overline{\{q_i \mid i \in \mathbb{N}\}} = K$.

For each $i \in \mathbb{N}$ the sequence $\{f_n(q_i)\}_{n=1}^{\infty}$ is bounded. Thus there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $\{f_{n_k}(q_1)\}_{k=1}^{\infty}$ is a convergent sequence of real numbers. For notational simplicity, denote this subsequence $\{f_{k,1}\}_{k=1}^{\infty}$. Proceeding inductively, we see that there exists a sequence of subsequences $\{f_n\}_{n=1}^{\infty} \supset \{f_{k,1}\}_{k=1}^{\infty} \supset \dots \supset \{f_{k,i}\}_{k=1}^{\infty} \supset \dots$ such that $\{f_{k,i}(q_i)\}_{k=1}^{\infty}$ is a convergent sequence of real numbers.

Consider now the subsequence $\{f_{j,j}\}_{j=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$. Note that for each subsequence $\{f_{k,i}\}$ there exists $J \in \mathbb{N}$ such that $\{f_{j,j}\}_{j=J}^{\infty} \subset \{f_{k,i}\}_{k=1}^{\infty}$. Thus $\{f_{j,j}(q_i)\}_{j=1}^{\infty}$ is a convergent sequence of real numbers for each $i \in \mathbb{N}$.

We claim that $\{f_{j,j}\}_{j=1}^{\infty}$ is a Cauchy sequence in $C^0(K)$. To see this, let $\varepsilon > 0$. The equicontinuity of $\{f_n\}$ implies that there exists $\delta > 0$ so that whenever $|x - y| < \delta$ we have $|f_{j,j}(x) - f_{j,j}(y)| < \varepsilon/3$ for all $j \in \mathbb{N}$. The compactness of K implies that there exists $I \in \mathbb{N}$ such that $\{B_\delta(q_i) \mid 1 \leq i \leq I\}$ is an open cover for K. The finiteness of this open cover and the convergence of the sequences $\{f_{j,j}(q_i)\}$ implies that there exists $J \in \mathbb{N}$ such that whenever $i \leq I$ and $j, j' \geq J$ we have $|f_{j,j}(q_i) - f_{j',j'}(q_i)| < \varepsilon/3$.

Fix $x \in K$. There exists $i \leq I$ such that $x \in B_\delta(q_i)$. Thus for $j, j' \geq J$ we have

$$\begin{aligned} &|f_{j,j}(x) - f_{j',j'}(x)| \\ &\quad \leq |f_{j,j}(x) - f_{j,j}(q_i)| + |f_{j,j}(q_i) - f_{j',j'}(q_i)| + |f_{j',j'}(q_i) - f_{j',j'}(x)| \\ &\qquad\qquad \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Taking the supremum over all $x \in K$ shows that $j, j' \geq J$ implies that $\|f_{j,j} - f_{j',f'}\|_{L^\infty(K)} < \varepsilon$. Thus $\{f_{j,j}\}$ is a Cauchy sequence. Since $C^0(K)$ is complete, the sequence converges to a continuous function $K \to \mathbb{R}$. □

Remark 2.32. *The Arzela-Ascoli theorem holds when K is any compact metric space. The proof relies on showing that any compact metric space has a countable dense subset. See, for example, the book* Principles of Mathematical Analysis *by Walter Rudin.*

Exercise 2.13. Find a subset $U \subset \mathbb{R}$ and a sequence of functions $f_n \colon U \to \mathbb{R}$ that is pointwise bounded but not uniformly bounded.