Chapter 1

Metric spaces

Definition 1.1. *A* ***metric d on set X*** *is a function* $d\colon X \times X \to \mathbb{R}$ *satisfying the following properties:*

- *Positive definite property: If* $x, y \in X$ *then* $d(x, y) \geq 0$*, with equality if and only if* $x = y$.
- *Symmetry property: If* $x, y \in X$ *then* $d(x, y) = d(y, x)$.
- *Triangle inequality: If* $x, y, z \in X$ *then* $d(x, y) \leq d(x, z) + d(z, y)$.

If d *is a metric on set* X*, we call* (X, d) *a* ***metric space****.*

Remark 1.2. *Often, especially if the metric does not appear explicitly in the formulas present, we simply write "*X *is a metric space" rather than "*(X, d) *is a metric space." In this case, the symbol "*X*" represents both the metric space and also the set* X*.*

Lemma 1.3. *The function* $d(x, y) = |x - y|$ *is a metric on the set* $\mathbb{R}$ *of real numbers.*

Remark 1.4. *Unless stated otherwise, when working with* $\mathbb{R}$ *we always make use of the metric* $d(x, y) = |x - y|$*, which we refer to as the "standard metric."*

1.1 Open and closed sets

Definition 1.5. *Suppose* (X, d) *is a metric space.*

1. *For each* $x \in X$ *and real number* $r > 0$*, define the* ***open ball*** $\boldsymbol{B_r(x)}$ *by*

$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$

2. *A set $U \subset X$ is* ***open in X*** *if for each $x \in U$ there exists $r > 0$ such that $B_r(x) \subset U$.*

3. *A set $V \subset X$ is* ***closed*** *if $X \setminus V$ is open.*

4. *A set $U \subset X$ is* ***bounded*** *if there exists some $x \in X$ and $r > 0$ such that $U \subset B_r(x)$.*

Proposition 1.6 (Basic properties of open sets)**.** *Let X be a metric space.*

1. *If $U, V \subset X$ are open then $U \cap V$ is open in X.*

2. *If for each $j \in J$ the set $U_j \subset X$ is open X, then $\bigcup_{j \in J} U_j$ is open. Here J is any (not necessarily countable) index set.*

3. *Both X and $\emptyset$ are open in X.*

Definition 1.7. *Suppose X is a metric space and $Y \subset X$.*

1. *$x \in X$ is a* ***boundary point of Y*** *if for each $r > 0$ the open ball $B_r(x)$ both contains an element of Y and contains an element of $X \setminus Y$. The collection of all boundary points is called the* ***boundary of Y*** *and is denoted ∂Y.*

2. *The* ***interior of Y*** *is the set $Y \setminus \partial Y$ and is denoted $\mathring{Y}$.*

3. *The* ***closure of Y*** *is the set $Y \cup \partial Y$ and is denoted $\overline{Y}$.*

Proposition 1.8. *Suppose X be a metric space and $Y \subset X$. Then*

1. *$\mathring{Y}$ is open,*

2. *$\overline{Y}$ is closed, and*

3. *∂Y is closed.*

Definition 1.9. *Let X be a metric space and $Y \subset X$. We say that* ***Y is dense in X*** *if $\overline{Y} = X$.*

Lemma 1.10. *Suppose (X, d) is a metric space and $Y \subset X$. Let $d_Y : Y \times Y \to \mathbb{R}$ be the restriction of d to Y. Then (Y, d_Y) is a metric space. The function d_Y is called the* ***induced metric*** *on Y.*

Definition 1.11. *Suppose (X, d) is a metric space and $V \subset Y \subset X$. We say a subset V is* ***open relative to Y*** *if V is open as a subset of (Y, d_Y).*

Lemma 1.12. *Suppose X is a metric space and $V \subset Y \subset X$. The set V is open relative to Y if and only if $V = Y \cap U$ for some open set $U \subset X$.*

Exercise 1.1. Let X be a metric space. Show that

1. if $U, V \subset X$ are closed then $U \cup V$ is closed in X,
2. if, for any index set J, for each $j \in J$ the set $U_j \subset X$ is closed then $\bigcap_{j \in J} U_j$ is closed, and
3. both X and $\emptyset$ are closed in X.

Find an example of an index set J and corresponding subsets $V_j \subset \mathbb{R}$ for which $\cup_{j \in J} V_j$ is not closed (relative to the standard metric).

Exercise 1.2. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

Exercise 1.3. Consider the metric space $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

Exercise 1.4. Consider the metric space $X = \{x \in \mathbb{R} \mid 1 < |x|\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < |x| \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

Exercise 1.5. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \mathbb{Q}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

1.2 Connectedness

Definition 1.13. *A metric space X is called* ***connected*** *if whenever $X = U_1 \sqcup U_1$ for open sets U_1, U_2 then either U_1 or U_2 is the empty set. (We write $X = U_1 \sqcup U_2$ when $X = U_1 \cup U_1$ and $U_1 \cap U_2 = \emptyset$.)*

A subset $Y \subset X$ is called ***connected*** *if Y is a connected metric space relative to the induced metric.*

Proposition 1.14. *Suppose X is a connected metric space and Y is a nonempty subset of X. If Y is both open and closed, then $Y = X$.*

Theorem 1.15 (Characterization of connected subsets of $\mathbb{R}$)**.** *Suppose $U \subset \mathbb{R}$. The following are equivalent:*

1. *U is connected.*
2. *If $a < b < c$ and $a, c \in U$, then $b \in U$.*

Exercise 1.6. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. Is X connected?
2. Is Y connected?

Exercise 1.7. Consider the metric space $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. Is X connected?
2. Is Y connected?

Exercise 1.8. Consider the metric space $X = \{x \in \mathbb{R} \mid 1 < |x|\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < |x| \leq 2\}$.

1. Is X connected?

2. Is Y connected?

Exercise 1.9. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \mathbb{Q}$.

1. Is Y connected?

1.3 Compact sets

Definition 1.16. *Let X be a metric space and $Y \subset X$.*

1. *An **open cover** of Y is a (possibly uncountable) collection $\{U_j\}_{j \in J}$ of open sets $U_j \subset X$ such that*

$$Y \subset \bigcup_{j \in J} U_j.$$

2. *Suppose $\{U_j\}_{j \in J}$ is an open cover of Y. If $J' \subset J$ is such that $\{U_j\}_{j \in J'}$ also covers Y then we say that $\{U_j\}_{j \in J'}$ is a **subcover** of $\{U_j\}_{j \in J}$.*

Definition 1.17. *Let X be a metric space. A set $K \subset X$ is called **compact** if every open cover of K has a finite subcover.*

*If X is a compact set, then we say that X is a **compact metric space**.*

Proposition 1.18. *Suppose X is a metric space and $K \subset X$ is compact. If Y is a closed subset of K, then Y is compact.*

Proposition 1.19. *The interval $[0, 1]$ is a compact subset of $\mathbb{R}$.*

Corollary 1.20. *Any interval $[a, b] \subset \mathbb{R}$ is compact.*

Exercise 1.10. Consider the metric space $\mathbb{R}$ (with the standard metric).

1. Is $\mathbb{R}$ compact?

2. Is the set $Y = \{x \in X \mid 1 < x \leq 2\}$ compact?

3. Is the set $\mathbb{Q}$ compact?

4. Make a conjecture about which subsets of $\mathbb{R}$ are compact and which are not.

1.4 Sequences in metric spaces

Definition 1.21. *Let X be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$* ***converges*** *to $x \in X$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$n \geq N \quad \Longrightarrow \quad d(x_n, x) < \varepsilon.$$

In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.22 (Open set criterion for convergence)**.** *The sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if and only if for each open set U containing x there exists $N \in \mathbb{N}$ such that*

$$n \geq N \quad \Longrightarrow \quad x_n \in U.$$

Proposition 1.23 (Sequential criterion for closedness)**.** *A subset U of X is closed if and only if for each sequence $\{x_n\}_{n=1}^{\infty} \subset U$ we have*

$$x_n \to x \quad \Longrightarrow \quad x \in U.$$

Definition 1.24. *Let X be a metric space and $K \subset X$. We say that K is* ***sequentially compact*** *if for each sequence $\{x_n\}_{n=1}^{\infty} \subset K$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to an element in K.*

Lemma 1.25 (Lebesgue number lemma)**.** *Let X be a metric space and suppose that K is sequentially compact. Suppose that $\{U_j\}_{j\in J}$ is an open cover of K. Then there exists $\delta > 0$ such that for each $x \in K$ there exists $j \in J$ such that $B_\delta(x) \subset U_j$. The number δ is called the* ***Lebesgue number*** *of the open cover.*

Proposition 1.26 (Sequential criterion for compactness)**.** *Let X be a metric space. A subset K of X is compact if and only if it is sequentially compact.*

1.5 Completeness and the real numbers

Definition 1.27. *A sequence $\{x_n\}_{n=1}^{\infty}$ is* ***Cauchy*** *if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$n, m \geq N \quad \Longrightarrow \quad d(x_n, x_m) < \varepsilon.$$

Proposition 1.28. *If a sequence is convergent, then it is Cauchy.*

Definition 1.29. *A metric space (X, d) is called* ***complete*** *if every Cauchy sequence converges to some element of X.*

Theorem 1.30. *Every compact metric space is complete.*

Theorem 1.31. *The real numbers, equipped with the metric*

$$d(x, y) = |x - y|,$$

form a complete metric space.

Theorem 1.32 (Bolzano-Weierstrass)**.** *Suppose* $\{x_n\}_{n=1}^{\infty}$ *is bounded sequence in* $\mathbb{R}$*. Then there exists a subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *that converges to some* $x \in \mathbb{R}$*.*

Theorem 1.33 (Heine-Borel)**.** *A set* $K \subset \mathbb{R}$ *is compact if and only if* K *is closed and bounded.*

1.6 Normed vector spaces

Definition 1.34. *Let* $\mathbb{V}$ *be a real vector space. A* ***norm*** *on* $\mathbb{V}$ *is a function* $\mathbb{V} \to \mathbb{R}$*, typically denoted by* $v \mapsto \|v\|$*, satisfying the following properties:*

1. *(Positive definite property) For each* $v \in \mathbb{V}$ *we have* $\|v\| \geq 0$*, with equality if and only if* $v = 0$*.*
2. *(Scaling property) For each* $v \in \mathbb{V}$ *and* $a \in \mathbb{R}$ *we have* $\|av\| = |a|\,\|v\|$*.*
3. *(Triangle inequality) For each* $v, w \in \mathbb{V}$ *we have* $\|v + w\| \leq \|v\| + \|w\|$*.*

When taken together with a norm, the vector space $\mathbb{V}$ *is called a* ***normed vector space****.*

Lemma 1.35. *Suppose* $\mathbb{V}$ *is a normed vector space with norm* $\|\cdot\|$*. Then the function* $d\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ *given by*

$$d(v, w) = \|v - w\|$$

is a metric on $\mathbb{V}$*.*

Definition 1.36. *A normed vector space that is complete is called a* ***Banach space****.*

Definition 1.37. *Let* $n \in \mathbb{N}$*.*

1. *The* ***dot product*** *is the function* $\mathbb{R}^n \times \mathbb{R}^n$ *given by*

$$\mathbf{x} \cdot \mathbf{y} = (x^1, \ldots, x^n) \cdot (y^1, \ldots, y^n) = \sum_{k=1}^{n} x^k y^k.$$

2. *The* ***Euclidean norm*** *on* $\mathbb{R}^n$ *is defined by* $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$.

Theorem 1.38. *For* $n \in \mathbb{N}$,

1. *the Euclidean norm is a norm on* $\mathbb{R}^n$,
2. *the Euclidean norm makes* $\mathbb{R}^n$ *a Banach space, and*
3. *the Heine-Borel and Bolzano-Weierstrass theorems hold for* $\mathbb{R}^n$.

1.7 Topological spaces

Definition 1.39. *Let* X *be any set. A collection* $\mathscr{T}$ *of subsets of* X *form a* ***topology*** *on* X *if*

1. $\emptyset, X \in \mathscr{T}$;
2. *if* $U, V \in \mathscr{T}$, *then* $U \cap V \in \mathscr{T}$; *and*
3. *for any index set* J, *if* $U_j \in \mathscr{T}$ *for all* $j \in J$ *then* $\bigcup_{j \in J} U_j \in \mathscr{T}$.

A set X, *together with a topology* $\mathscr{T}$ *on* X, *is called a* ***topological space***.

Lemma 1.40. *If* X *is a metric space, then the collection of open subsets of* X *forms a topology on* X.

Definition 1.41. *Suppose both* d_1 *and* d_2 *are metrics on* X. *We say that* d_1 *is* ***topologically equivalent*** *to* d_2 *if the two metrics give rise to the same topology (i.e. give rise to the same collection of open sets).*

Lemma 1.42 (Interior ball lemma)**.** *Suppose both* d_1 *and* d_2 *are topologically equivalent metrics on* X. *If* $U \subset X$ *is open with respect to* d_1, *then for any* $x \in U$ *there exists* $r > 0$ *such that*

$$\{y \in X \mid d_2(x, y) < r\} \subset U.$$

Proposition 1.43 (Sequential criterion for equivalence)**.** *Two metrics on* X, d_1 *and* d_2, *are topologically equivalent if and only if sequences converges with respect to* d_1 *if and only if they converges with respect to* d_2.

Definition 1.44. *We say that a definition regarding metric spaces is* ***topological*** *if the collection of objects satisfying the definition do not change when the metrics present are replaced by topologically equivalent metrics.*

Theorem 1.45. *The following definitions are topological:*

1. *interior, boundary, and closure of a set;*
2. *compactness of a set; and*
3. *connectedness of a set.*