Chapter 1

Metric spaces

Definition 1.1. *A* ***metric d on set X*** *is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following properties:*

- *Positive definite property: If $x, y \in X$ then $d(x, y) \geq 0$, with equality if and only if $x = y$.*
- *Symmetry property: If $x, y \in X$ then $d(x, y) = d(y, x)$.*
- *Triangle inequality: If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.*

If d is a metric on set X, we call (X, d) a ***metric space****.*

Remark 1.2. *Often, especially if the metric does not appear explicitly in the formulas present, we simply write "X is a metric space" rather than "(X, d) is a metric space." In this case, the symbol "X" represents both the metric space and also the set X.*

Lemma 1.3. *The function $d(x, y) = |x - y|$ is a metric on the set $\mathbb{R}$ of real numbers.*

Proof. This was done in the Real Analysis course. □

Remark 1.4. *Unless stated otherwise, when working with $\mathbb{R}$ we always make use of the metric $d(x, y) = |x-y|$, which we refer to as the "standard metric."*

1.1 Open and closed sets

Definition 1.5. *Suppose (X, d) is a metric space.*

1. *For each* $x \in X$ *and real number* $r > 0$*, define the* ***open ball*** $\boldsymbol{B_r(x)}$ *by*
$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$
2. *A set* $U \subset X$ *is* ***open in X*** *if for each* $x \in U$ *there exists* $r > 0$ *such that* $B_r(x) \subset U$*.*
3. *A set* $V \subset X$ *is* ***closed*** *if* $X \setminus V$ *is open.*
4. *A set* $U \subset X$ *is* ***bounded*** *if there exists some* $x \in X$ *and* $r > 0$ *such that* $U \subset B_r(x)$*.*

Proposition 1.6 (Basic properties of open sets)**.** *Let* X *be a metric space.*

1. *If* $U, V \subset X$ *are open then* $U \cap V$ *is open in* X*.*
2. *If for each* $j \in J$ *the set* $U_j \subset X$ *is open* X*, then* $\bigcup_{j \in J} U_j$ *is open. Here* J *is any (not necessarily countable) index set.*
3. *Both* X *and* $\emptyset$ *are open in* X*.*

Proof. Simple element chasing. □

Definition 1.7. *Suppose* X *is a metric space and* $Y \subset X$*.*

1. $x \in X$ *is a* ***boundary point of Y*** *if for each* $r > 0$ *the open ball* $B_r(x)$ *both contains an element of* Y *and contains an element of* $X \setminus Y$*. The collection of all boundary points is called the* ***boundary of Y*** *and is denoted* ∂Y*.*
2. *The* ***interior of Y*** *is the set* $Y \setminus \partial Y$ *and is denoted* $\mathring{Y}$*.*
3. *The* ***closure of Y*** *is the set* $Y \cup \partial Y$ *and is denoted* $\overline{Y}$*.*

Proposition 1.8. *Suppose* X *is a metric space and* $Y \subset X$*. Then*

1. $\mathring{Y}$ *is open,*
2. $\overline{Y}$ *is closed, and*
3. ∂Y *is closed.*

Proof.

1. Let $x \in \mathring{Y}$. Since $x \notin \partial Y$ there exists $r > 0$ such that either $B_r(x) \subset Y$ or $B_r(x) \subset (X \setminus Y)$. The latter is excluded because $x \in Y$; thus $B_r(x) \subset Y$.

   Suppose, however, that for each $\varepsilon \in (0, r]$ there exists $y \in B_\varepsilon(x) \cap \partial Y$. By the definition of ∂Y there exists $x \in B_\varepsilon(y) \setminus Y$. Thus by the triangle inequality, $z \in B_{2\varepsilon}(x)$. Taking $\varepsilon = r/2$ yields a contradiction. Thus there must exist some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \partial Y = \emptyset$. Thus $B_\varepsilon(x) \subset \mathring{Y}$ and $\mathring{Y}$ is open.

2. Let $x \in X \setminus \overline{Y}$ and suppose that for all $r > 0$ there exists $y \in B_r(x) \cap \overline{Y}$. Since $y \in \overline{Y} = Y \cup \partial Y$, there exists $z \in B_r(y) \cap Y$. By triangle inequality we have $z \in B_{2r}(x) \cap Y$. Consequently, we see that for each $r > 0$ the ball $B_r(x)$ contains a point in Y and also contains $x \notin Y$. Thus $x \in \partial Y$, which is a contradiction. Therefore there must exist some $r > 0$ such that $B_r(x) \subset X \setminus \overline{Y}$. This means that $X \setminus \overline{Y}$ is open and therefore that $\overline{Y}$ is closed.

3. The basic properties of open sets shows that $\mathring{Y} \cup (X \setminus \overline{Y})$ is open. Therefore the complement of this set, which is the same as ∂Y, is closed.

□

Definition 1.9. *Let X be a metric space and $Y \subset X$. We say that **Y is dense in X** if $\overline{Y} = X$.*

Lemma 1.10. *Suppose (X, d) is a metric space and $Y \subset X$. Let $d_Y : Y \times Y \to \mathbb{R}$ be the restriction of d to Y. Then (Y, d_Y) is a metric space. The function d_Y is called the **induced metric** on Y.*

Proof. The required properties of d_Y follow immediate from the fact that they hold for all elements of X and the fact that $Y \subset X$. □

Definition 1.11. *Suppose (X, d) is a metric space and $V \subset Y \subset X$. We say a subset V is **open relative to Y** if V is open as a subset of (Y, d_Y).*

Lemma 1.12. *Suppose X is a metric space and $V \subset Y \subset X$. The set V is open relative to Y if and only if $V = Y \cap U$ for some open set $U \subset X$.*

Proof. Suppose $V \subset Y$ is open. Then for all $v \in V$ there exists $r(v) > 0$ such that $B_{r(v)}(v) \cap Y \subset V$. But

$$V = \bigcup_{v \in V} (B_{r(v)}(v) \cap Y) = \left(\bigcup_{v \in V} B_{r(v)}(v) \right) \cap Y,$$

which is an open subset of X intersected with Y.

If $V = U \cap Y$ for some open $U \subset X$, then the openness of U implies that for each $v \in V$ there exists $r > 0$ such that $B_r(v) \subset U$. Hence $B_r(v) \cap Y \subset V$, and thus V is open relative to Y. $\square$

Exercise 1.1. Let X be a metric space. Show that

1. if $U, V \subset X$ are closed then $U \cup V$ is closed in X,
2. if, for any index set J, for each $j \in J$ the set $U_j \subset X$ is closed then $\bigcap_{j \in J} U_j$ is closed, and
3. both X and $\emptyset$ are closed in X.

Find an example of an index set J and corresponding closed subsets $V_j \subset \mathbb{R}$ for which $\cup_{j \in J} V_j$ is not closed (relative to the standard metric).

Exercise 1.2. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

Exercise 1.3. Consider the metric space $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?
4. Is Y open (relative to X), closed, or neither?

Exercise 1.4. Consider the metric space $X = \{x \in \mathbb{R} \mid 1 < |x|\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < |x| \leq 2\}$.

1. What is ∂Y?
2. What is $\mathring{Y}$?
3. What is $\overline{Y}$?

4. Is Y open (relative to X), closed, or neither?

Exercise 1.5. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \mathbb{Q}$.

1. What is ∂Y?

2. What is $\mathring{Y}$?

3. What is $\overline{Y}$?

4. Is Y open (relative to X), closed, or neither?

1.2 Connectedness

Definition 1.13. *A metric space X is called **connected** if whenever $X = U_1 \sqcup U_1$ for open sets U_1, U_2 then either U_1 or U_2 is the empty set. (We write $X = U_1 \sqcup U_2$ when $X = U_1 \cup U_1$ and $U_1 \cap U_2 = \emptyset$.)*

*A subset $Y \subset X$ is called **connected** if Y is a connected metric space relative to the induced metric.*

Proposition 1.14. *Suppose X is a connected metric space and Y is a nonempty subset of X. If Y is both open and closed, then $Y = X$.*

Proof. If Y is both open and closed, then $X = Y \sqcup (X \setminus Y)$ is the disjoint union of two open sets. By connectedness of X, one of the two sets must be empty. Since Y is nonempty, we must have $X \setminus Y = \emptyset$, meaning $X = Y$. □

Theorem 1.15 (Characterization of connected subsets of $\mathbb{R}$)**.** *Suppose $U \subset \mathbb{R}$. The following are equivalent:*

1. *U is connected.*

2. *If $a < b < c$ and $a, c \in U$, then $b \in U$.*

Proof. To show the first condition implies the second we proceed by contradiction. Suppose U is connected, that $a, c \in U$, that $a < b < c$, and that $c \notin U$. Let $A = U \cap (-\infty, b)$ and $C = U \cap (b, \infty)$. Both A and C are open relative to U and are both nonempty, containing a and c, respectively. But $U = A \sqcup C$, which contradicts the connectedness of U.

To show the second condition implies the first, we show the contrapositive. Suppose now that U is not connected, so that $U = A \sqcup C$, where A and C are nonempty, open sets. Let $a \in A$ and $c \in C$. Without loss of generality $a < c$. Let $b = \sup\{x \in A \mid a \leq x \leq b\}$. If $b \in A$, then by A being open there exists $r > 0$ such that $(b - r, b + r) \subset A$, which contradicts the supremum property of b. Likewise, if $b \in C$, then there exists $r > 0$ such that $(b - r, b + r) \subset C$, which again contradicts the defining property of b. Thus we conclude that $b \notin A \cup B = U$. □

Exercise 1.6. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. Is X connected?

2. Is Y connected?

Exercise 1.7. Consider the metric space $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < x \leq 2\}$.

1. Is X connected?

2. Is Y connected?

Exercise 1.8. Consider the metric space $X = \{x \in \mathbb{R} \mid 1 < |x|\}$ with the induced metric $d(x, y) = |x - y|$ and let $Y = \{x \in X \mid 1 < |x| \leq 2\}$.

1. Is X connected?

2. Is Y connected?

Exercise 1.9. Consider the metric space $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ and let $Y = \mathbb{Q}$.

1. Is Y connected?

1.3 Compact sets

Definition 1.16. *Let X be a metric space and $Y \subset X$.*

1. *An **open cover** of Y is a (possibly uncountable) collection $\{U_j\}_{j \in J}$ of open sets $U_j \subset X$ such that*
$$Y \subset \bigcup_{j \in J} U_j.$$

2. *Suppose $\{U_j\}_{j \in J}$ is an open cover of Y. If $J' \subset J$ is such that $\{U_j\}_{j \in J'}$ also covers Y then we say that $\{U_j\}_{j \in J'}$ is a **subcover** of $\{U_j\}_{j \in J}$.*

Definition 1.17. *Let X be a metric space. A set $K \subset X$ is called **compact** if every open cover of K has a finite subcover.*

*If X is a compact set, then we say that X is a **compact metric space**.*

Proposition 1.18. *Suppose K is a compact metric space. If Y is a closed subset of K, then Y is compact.*

Proof. Suppose K is compact and $Y \subset K$ is closed. Let $\{U_j\}_{j\in J}$ be an open cover of Y. Since Y is closed, $V = X \setminus Y$ is open. Furthermore, $\{V\} \cup \{U_j\}_{j\in J}$ is an open cover of K and thus admits a finite subcover. Removing V from the finite subcover, if necessary, yields a finite subcover of Y. □

Proposition 1.19. *The interval $[0, 1]$ is a compact subset of $\mathbb{R}$.*

Proof. The proof is by contradiction. Suppose $\{U_j\}_{j\in J}$ is an open cover of $[0, 1]$ having no finite subcover. We show inductively that this implies the existence of numbers $\{a_n, b_n\}_{n=0}^{\infty} \subset [0, 1]$ having the properties:

- $b_n - a_n = 2^{-n}$, and
- the interval $[a_n, b_n]$ cannot be covered by a finite subcover of $\{U_j\}_{j\in J}$.

For the base case, we simply set $a_n = 0$ and $b_n = 1$. For the inductive step, we consider the interval $[a_n, b_n] = [a_n, c_n] \cup [c_n, b_n]$, where $c_n = (a_n + b_n)/2$. It must be the case that either the subinterval $[a_n, c_n]$ or the subinterval $[c_n, b_n]$ cannot be covered by a finite subcover of $\{U_j\}_{j\in J}$. We define a_{n+1} and b_{n+1} by the condition that $[a_{n+1}, b_{n+1}]$ is this subinterval.

Set $a = \sup\{a_n\}$ and $b = \inf\{b_n\}$. Since $a_n \le a_{n+1} < b_{n+1} \le b_n$ for all n we easily have $a_n \le a \le b \le b_n$ for all n. But $b_n - a_n = 2^{-n}$ and thus $a = b$.

Since $\{U_j\}_{j\in J}$ covers $[0, 1]$ there exists some $\hat{\jmath} \in J$ such that $a \in U_{\hat{\jmath}}$. By $U_{\hat{\jmath}}$ being open, there exists $r > 0$ such that $(a - r, a + r) \subset U_{\hat{\jmath}}$. But there exists positive integer N such that $2^{-N} < r/2$. For such a N we have $a \in [a_N, b_N]$ and $b_N - a_N < r/2$. Thus $[a_N, b_N] \subset (a - r, a + r) \subset U_{\hat{\jmath}}$, which contradicts the property of $[a_N, b_N]$ that it cannot be covered by a subcover of $\{U_j\}_{j\in J}$. □

Corollary 1.20. *Any interval $[a, b] \subset \mathbb{R}$ is compact.*

Exercise 1.10. Consider the metric space $\mathbb{R}$ (with the standard metric).

1. Is $\mathbb{R}$ compact?
2. Is the set $Y = \{x \in X \mid 1 < x \le 2\}$ compact?

3. Is the set $\mathbb{Q}$ compact?

4. Make a conjecture about which subsets of $\mathbb{R}$ are compact and which are not.

1.4 Sequences in metric spaces

Definition 1.21. *Let X be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$* ***converges*** *to $x \in X$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$n \geq N \quad \Longrightarrow \quad d(x_n, x) < \varepsilon.$$

In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.22 (Open set criterion for convergence)**.** *The sequence $\{x_n\}_{n=1}^{\infty}$ converges to x if and only if for each open set U containing x there exists $N \in \mathbb{N}$ such that*

$$n \geq N \quad \Longrightarrow \quad x_n \in U.$$

Proof. Suppose $x_n \to x$ and and let U be an open set containing x. By definition of open, there exists $r > 0$ such that $B_r(x) \subset U$. By definition of convergence, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < r$. Thus for all $n \geq N$ we have $x_n \in B_r(x) \subset U$.

To show the converse, consider any $\varepsilon > 0$. By hypothesis, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_\varepsilon(x)$. Thus $x_n \to x$. □

Proposition 1.23 (Sequential criterion for closedness)**.** *A subset U of X is closed if and only if for each sequence $\{x_n\}_{n=1}^{\infty} \subset U$ we have*

$$x_n \to x \quad \Longrightarrow \quad x \in U.$$

Proof. Suppose that $U \subset X$ is closed, than $\{x_n\}_{n=1}^{\infty} \subset U$, that $x_n \to x$, but that $x \notin U$. Since U is closed, $X \setminus U$ is open. Thus there exists $r \geq 0$ such that $B_r(x) \subset X \setminus U$. Consequently, we must have $d(x_n, x) \geq r/2$ for all n, which contradicts $x_n \to x$.

To show the converse we assume that $U \subset X$ is not closed, meaning that $X \setminus U$ is not open. This means that there exists $x \in X \setminus U$ having the property that for each $n \in \mathbb{N}$ there exists some $x_n \in B_{1/n}(x) \cap U$. This implies that $\{x_n\}_{n=1}^{\infty} \subset U$ and that $x_n \to x$, but that $x \notin U$, which is our desired contradiction. □

Definition 1.24. *Let X be a metric space and $K \subset X$. We say that K is* ***sequentially compact*** *if for each sequence $\{x_n\}_{n=1}^{\infty} \subset K$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to an element in K.*

Lemma 1.25 (Lebesgue number lemma)**.** *Let X be a metric space and suppose that K is sequentially compact. Suppose that $\{U_j\}_{j\in J}$ is an open cover of K. Then there exists $\delta > 0$ such that for each $x \in K$ there exists $j \in J$ such that $B_\delta(x) \subset U_j$. The number δ is called the* ***Lebesgue number*** *of the open cover.*

Proof. Suppose K is sequentially compact and that $\{U_j\}_{j\in J}$ is an open cover of K. We proceed by contradiction, and thus assume that for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that $B_{1/n}(x_n)$ is not contained in any set U_j.

By sequential compactness, there exists a subsequence x_{n_k} converging to some $x \in K$. Since $\{U_j\}$ is an open cover of K, there exists $\hat{\jmath}$ such that $x \in U_{\hat{\jmath}}$. Since $U_{\hat{\jmath}}$ is open, there exists $r > 0$ such that $B_r(x) \subset U_{\hat{\jmath}}$.

By convergence, there exists $N \in \mathbb{N}$ such that $k \geq N$ implies that $x_{n_k} \in B_{r/2}(x)$. As $n_k \to \infty$, we may choose N sufficiently large that $k \geq N$ implies also that $n_k > 2/r$. Thus for $k \geq N$ we have $B_{1/n_k}(x_{n_k}) \subset B_r(x) \subset U_{\hat{\jmath}}$, which is our desired contradition. □

Proposition 1.26 (Sequential criterion for compactness)**.** *Let X be a metric space. A subset K of X is compact if and only if it is sequentially compact.*

Proof. We first prove that compactness implies sequential compactness. We proceed by contradiction and suppose that K is compact and that there exists $\{x_n\}_{n=1}^{\infty} \subset K$ having no convergence subsequence.

Let $Z = \{x_n \mid n \in \mathbb{N}\}$. If $Z \subset X$ is not closed, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset Z$ such that $y_k \to y$ with $y \notin Z$. Since $d(y, x_n) > 0$ for all n, the definition of convergence implies that each $x_n \in Z$ can appear at most a finite number of times in the sequence $\{y_k\}$. Thus there exists a subsequence of $\{y_k\}$, which we call $\{z_k\}$, such that $z_k \to y$ and each z_k is unique. For each element z_k in this sequence, there exists $n_k \in \mathbb{N}$ such that $z_k = x_{n_k}$. Working inductively, we may extract a subsequence of $\{z_k\}$ for which the corresponding sequence n_k is increasing. The result is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to y. As this contradicts our assumption that $\{x_n\}$ has no convergence subsequence, we see that Z must be closed. Consequently, the set $U = X \setminus Z$ is open.

Since $\{x_n\}$ has no convergent subsequences, each point $x \in K$ appears at most a finite number of times in the sequence. Removing any repeated points in the sequence yields a subsequence $\{\tilde{x}_k\}_{k=1}^{\infty}$ of $\{x_n\}$ such that $k \neq l$

implies $\tilde{x}_k \neq \tilde{x}_l$ and such that $Z = \{\tilde{x}_k \mid k \in \mathbb{N}\}$. Notice that $\{\tilde{x}_k\}$ also has no convergence subsequences. This implies that for each $k \in \mathbb{N}$ there exists $r_k > 0$ such that $Z \cap B_{r_k}(\tilde{x}_k) = \{\tilde{x}_k\}$.

Notice that the open sets $B_{r_k}(\tilde{x}_k)$, together with V, form an open cover of X and thus of K. Any finite subcover of this open cover cannot cover K, since at least one of the sets of the form $B_{r_k}(\tilde{x}_k)$ will be excluded and thus $\tilde{x}_k$ will not be contained in the subcover. This contradicts our assumption that K is compact.

We now assume that K is sequentially compact and show by contradiction that K is compact. Suppose $\{U_j\}_{j\in J}$ is an open cover of K having no finite subcover. Let δ be the Lebesgue number of this open cover.

We inductively construct a sequence $\{x_k\}_{k=1}^{\infty} \subset K$ such that for each $k \in \mathbb{N}$ we have $d(x_k, x_l) \geq \delta$ for all $l < k$. For the base case, we simply let x_1 be any point in K. Suppose now that we have a sequence $\{x_k\}_{k=1}^{n} \subset K$ satisfying the desired condition. The Lebesgue number lemma implies that for each $k = 1, \ldots, N$ there exists $j_k \in J$ such that $B_\delta(x_k) \subset U_{j_k}$. Since the open cover $\{U_j\}$ has no finite subcover, there exists $x_{N+1} \in K \setminus \bigcup_{k=1}^{N} U_{j_k}$, which completes the inductive step.

Consider now the sequence $\{x_k\}_{k=1}^{\infty}$. Since K is sequentially compact, there must exist a subsequence converging to some $x \in K$. Thus there must exist $k_1, k_2 \in \mathbb{N}$ such that $x_{k_1}, x_{k_2} \in B_{\delta/2}(x)$. The triangle inequality implies that $d(x_{k_1}, x_{k_2}) < \delta$, which is a contradiction. Thus it must be that the open cover $\{U_j\}$ contains a finite subcover; hence K is compact. □

1.5 Completeness and the real numbers

Definition 1.27. *A sequence $\{x_n\}_{n=1}^{\infty}$ is* ***Cauchy*** *if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$n, m \geq N \quad \Longrightarrow \quad d(x_n, x_m) < \varepsilon.$$

Proposition 1.28. *If a sequence is convergent, then it is Cauchy.*

Proof. Suppose $x_n \to x$ and let $\varepsilon > 0$. By definition, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $d(x_n, x) < \varepsilon/2$. Thus if $n, m \geq N$ the triangle inequality implies that $d(x_n, x_m) < \varepsilon$. □

Definition 1.29. *A metric space (X, d) is called* ***complete*** *if every Cauchy sequence converges to some element of X.*

Theorem 1.30. *Every compact metric space is complete.*

Proof. Let X be a compact metric space and $\{x_n\}_{n=1}^{\infty} \subset X$ be a Cauchy sequence. Since X is compact, it is also sequentially compact and thus there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x \in X$.

Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ such that $m, n \geq N$ implies $d(x_n, x_m) < \varepsilon/2$ and such that $n_k \geq N$ implies $d(x, x_{n_k}) < \varepsilon/2$. Thus by the triangle inequality $n \geq N$ implies $d(x, x_n) < \varepsilon$. Hence $x_n \to x \in X$ and X is complete. □

Theorem 1.31. *The real numbers, equipped with the metric*

$$d(x, y) = |x - y|,$$

form a complete metric space.

Proof. Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a Cauchy sequence. The definition of a sequence being Cauchy implies that the sequence in bounded, and is thus a sequence in some interval $[a, b]$. Since $[a, b]$ is compact, and since compact metric spaces are complete, the sequence converges to some $x \in [a, b] \subset \mathbb{R}$. □

Theorem 1.32 (Bolzano-Weierstrass)**.** *Suppose $\{x_n\}_{n=1}^{\infty}$ is bounded sequence in $\mathbb{R}$. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to some $x \in \mathbb{R}$.*

Proof. If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, then it is a sequence in some interval $[a, b]$. Since $[a, b]$ is compact, and thus sequentially compact, the sequence has a subsequence converging in $[a, b]$, and hence in $\mathbb{R}$. □

Theorem 1.33 (Heine-Borel)**.** *A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.*

Proof. Suppose $K \subset \mathbb{R}$ is compact. If $\{x_n\}_{n=1}^{\infty} \subset K$ with $x_n \to x$, then the sequential compactness of K implies that $x \in K$ and thus K is closed by the sequential criterion for closedness. Furthermore, the collection $\{B_n(0)\}_{n \in \mathbb{N}}$ provides an open cover for K that, if K is not bounded, has no finite subcover. Thus K is bounded.

Suppose now that K is closed and bounded. Let $\{x_n\}_{n=1}^{\infty} \subset K$. Since K is bounded, there exists interval $r > 0$ such that $K \subset [-r, r]$. Since the interval $[-r, r]$ is compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to some $x \in [-r, r]$. But K is closed, and thus $x_{n_k} \to x$ implies $x \in K$, from which we conclude that K is sequentially compact, and thus compact. □

1.6 Normed vector spaces

Definition 1.34. *Let $\mathbb{V}$ be a real vector space. A* ***norm*** *on $\mathbb{V}$ is a function $\mathbb{V} \to \mathbb{R}$, typically denoted by $v \mapsto \|v\|$, satisfying the following properties:*

1. *(Positive definite property) For each $v \in \mathbb{V}$ we have $\|v\| \geq 0$, with equality if and only if $v = 0$.*
2. *(Scaling property) For each $v \in \mathbb{V}$ and $a \in \mathbb{R}$ we have $\|av\| = |a| \, \|v\|$.*
3. *(Triangle inequality) For each $v, w \in \mathbb{V}$ we have $\|v + w\| \leq \|v\| + \|w\|$.*

When taken together with a norm, the vector space $\mathbb{V}$ is called a ***normed vector space****.*

Lemma 1.35. *Suppose $\mathbb{V}$ is a normed vector space with norm $\| \cdot \|$. Then the function $d \colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ given by*

$$d(v, w) = \|v - w\|$$

is a metric on $\mathbb{V}$.

Proof. The result follows immediately from the definition of a norm. □

Definition 1.36. *A normed vector space that is complete is called a* ***Banach space****.*

Definition 1.37. *Let $n \in \mathbb{N}$.*

1. *The* ***dot product*** *is the function $\mathbb{R}^n \times \mathbb{R}^n$ given by*

$$\mathbf{x} \cdot \mathbf{y} = (x^1, \ldots, x^n) \cdot (y^1, \ldots, y^n) = \sum_{k=1}^{n} x^k y^k.$$

2. *The* ***Euclidean norm*** *on $\mathbb{R}^n$ is defined by $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$.*

Theorem 1.38. *For $n \in \mathbb{N}$,*

1. *the Euclidean norm is a norm on $\mathbb{R}^n$,*
2. *the Euclidean norm makes $\mathbb{R}^n$ a Banach space, and*
3. *the Heine-Borel and Bolzano-Weierstrass theorems hold for $\mathbb{R}^n$.*

Proof. We sketch the proof, leaving details to the student.

1. The positive definite and scaling properties are immediate. The Cauchy-Schwartz inequality implies that $\|\mathbf{x}\cdot\mathbf{y}\| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$. Thus

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

   The triangle inequality immediately follows.

2. The completeness of $\mathbb{R}^n$ follows from the completeness of $\mathbb{R}$.

3. The Heine-Borel and Bolzano-Weierstrass theorems follow from the corresponding theorems in $\mathbb{R}$ once we have established the compactness of sets of the form

$$[a_1, b_1] \times \cdots \times [a_n, b_n],$$

   which we call "rectangular cells." The proof of this fact follows by adapting the "nested intervals" proof used to show that $[0, 1]$ is compact; in the adaptation one simply constructs a sequence of "nested rectangular cells," which have no finite subcover. □

1.7 Topological spaces

Definition 1.39. *Let X be any set. A collection $\mathscr{T}$ of subsets of X form a* ***topology*** *on X if*

1. *$\emptyset, X \in \mathscr{T}$;*
2. *if $U, V \in \mathscr{T}$, then $U \cap V \in \mathscr{T}$; and*
3. *for any index set J, if $U_j \in \mathscr{T}$ for all $j \in J$ then $\bigcup_{j\in J} U_j \in \mathscr{T}$.*

A set X, together with a topology $\mathscr{T}$ on X, is called a ***topological space****.*

Lemma 1.40. *If X is a metric space, then the collection of open subsets of X forms a topology on X.*

Proof. These are precisely the basic properties of open sets proved earlier. □

Definition 1.41. *Suppose both d_1 and d_2 are metrics on X. We say that d_1 is* ***topologically equivalent*** *to d_2 if the two metrics give rise to the same topology (i.e. give rise to the same collection of open sets).*

Lemma 1.42 (Interior ball lemma)**.** *Suppose both d_1 and d_2 are topologically equivalent metrics on X. If $U \subset X$ is open with respect to d_1, then for any $x \in U$ there exists $r > 0$ such that*

$$\{y \in X \mid d_2(x, y) < r\} \subset U.$$

Proof. This follows immediately from the definitions of open and topologically equivalent. □

Proposition 1.43 (Sequential criterion for equivalence)**.** *Two metrics on X, d_1 and d_2, are topologically equivalent if and only if sequences converges with respect to d_1 if and only if they converges with respect to d_2.*

Proof. The forward direction follows immediately from the interior ball lemma and the open set criterion for convergence.

The converse follows from the observation that if sequential convergence with respect to d_1 is equivalent to convergence with respect to d_2, then by the sequential criterion for closedness the two metrics give rise to the same closed sets. □

Definition 1.44. *We say that a definition regarding metric spaces is* ***topological*** *if the collection of objects satisfying the definition do not change when the metrics present are replaced by topologically equivalent metrics.*

Theorem 1.45. *The following definitions are topological:*

1. *interior, boundary, and closure of a set;*
2. *compactness of a set; and*
3. *connectedness of a set.*

Proof. Suppose both d_1 and d_2 are topologically equivalent metrics on X. Denote open balls defined with respect to d_1 by $B^1_r(x)$, and open balls defined with respect to d_2 by $B^2_r(x)$.

1. Let $Y \subset X$. Let y be a boundary point of Y with respect to d_1. Fix $r > 0$ and consider the open ball $B^2_r(y)$. By the interior ball lemma, there exists $B^1_s(y) \subset B^2_r(y)$. Since y is a boundary point of Y with respect to d_1 the intersection of $B^1_s(y)$, and hence $B^2_r(y)$, with both Y and $X \setminus Y$ are nonempty. Thus y is a boundary point of Y with respect to d_2.

   Since ∂Y is toplogical, it immediately follows that $\mathring{Y}$ and $\overline{Y}$ are topological.

2. Suppose K is compact with respect to d_1. Thus any sequence in K has a convergence subsequence that converges with respect to d_1. But if a subsequence converges with respect to d_1, it converges with respect to d_2. Consequently, K is sequentially compact, and thus compact, with respect to d_2.

3. This follows immediately from the definitions. □