# Homework regarding integration

Math 442, Spring 2017

Due: 22 March

## 1 The Cauchy integral remainder formula (corrected version)

Consider an interval $(-a, a) \subset \mathbb{R}$ and assume that $f\colon (-a, a) \to \mathbb{R}$ is in $C^k((-a, a))$ for all $k \in \mathbb{N}$. Let $x \in (-a, a)$.

1. Explain why
$$f(x) = f(0) + \int_0^x f'(t)\, dt.$$

2. Use integration by parts to show that
$$f(x) = f(0) + f'(0)\, x + \int_0^x f''(t)\, (x - t)\, dt.$$

3. Use induction to show that for all $n \in \mathbb{N}$ we have
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k + \frac{1}{n!} \int_0^x f^{(n+1)}(t)\, (x - t)^n\, dt.$$

   This formula is known as the *Cauchy integral remainder formula.*

4. Suppose $f(x) = \sin(x)$. What does the formula yield with $n = 4, 8$ at $x = \frac{\pi}{2}$?

## 2 A slice of $\pi$

For each $n \in \mathbb{N}$ define the functions $f_n, R_n\colon [0, 1] \to \mathbb{R}$ by
$$f_n(t) = \sum_{k=0}^{n} (-1)^n x^{2n} \quad \text{and} \quad R_n(t) = \frac{(-1)^{n+1} x^{2n+2}}{1 + t^2}.$$

1. Use the partial sum formula for geometric series to show that for each $n \in \mathbb{N}$ we have
$$\frac{1}{1 + t^2} = f_n(t) + R_n(t).$$

2. Show that for all $n \in \mathbb{N}$ we have
$$\int_0^1 R_n(t)\, dt \le \frac{1}{2n + 3}.$$

3. Show that for all $n \in \mathbb{N}$ we have

$$\int_0^1 f_n(t)\,dt = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}.$$

4. Use methods from Calculus II to compute

$$\int_0^1 \frac{1}{1+t^2}\,dt.$$

5. Explain how we may conclude that

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

# 3 An interesting sequence of functions

Consider the sequence of functions $\{f_n \colon [-1,1] \to \mathbb{R}\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \begin{cases} n(1-|x|) & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{if } |x| > \frac{1}{n} \end{cases}$$

1. Sketch the graph of a 'typical' $f_n$.

2. Show that for each $n \in \mathbb{N}$ the function $f_n$ is integrable.

3. Compute

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(t)\,dt.$$

4. Explain why Proposition 3.37 does not apply in this situation.

# Optional Challenge

1. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is integrable on every interval $[a,b] \subset \mathbb{R}$. Fix $a, I \in \mathbb{R}$. Give a definition of what it would mean for

$$\int_a^{\infty} f(x)\,dx$$

to converge to $I$.

2. Prove the "integral test": Suppose $f \in C^0([0,\infty))$ and is monotonic decreasing with $\lim_{x\to\infty} f(x) = 0$. Then the integral

$$\int_1^{\infty} f(x)\,dx$$

and the series

$$\sum_{k=1}^{\infty} f(k)$$

either both converge or both diverge.