Equilibrium solutions to the logistic model in the case of constant forcing

Paul Allen

Fall 2016

Abstract

We consider the logistic model for population growth together with a constant forcing term. We analyze situations under which the model admits equilibrium solutions.

1 Introduction

The logistic model describes the growth of a population that is subject to habitat constraint by assuming that the relative rate of change of the population is proportional to the percent of available habitat. We express this assumption as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \tag{1}$$

where the function P describes the size of the population at time t, the carrying capacity K is the population that can be sustainably supported by the habitat, and the proportionality constant r may be interpreted as the relative growth rate under ideal circumstances. We modify (1) by including a constant forcing term; the resulting model is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) + h. \tag{2}$$

We assume that $r > 0$ and $K > 0$, but do not make any assumptions on the sign of h.

Equilibrium, or constant, solutions to (2) represent population sizes that are unchanged according to the model. The analysis below derives conditions on the parameters r, K, and h under which the model (2) admits equilibrium solutions. We also provide interpretation of the conditions we develop.

2 Equilibrium solutions

In order for P to be a constant solution to (2) we must have

$$0 = rP\left(1 - \frac{P}{K}\right) + h.$$

Applying the quadratic formula we see that the equilibrium solution are given by

$$P = \frac{K}{2} \pm \frac{K}{2}\sqrt{1 + \frac{4h}{Kr}}. \tag{3}$$

Examining (3) we see that in order for there to be an equilibrium solution the quantity $1 + \frac{4h}{Kr}$ must be nonnegative. It is convenient to express this condition as

$$h \geq -\frac{Kr}{4}. \tag{4}$$

Notice that the quantity Kr has units of population per time, and is thus a rate. We can understand the (4) as a lower bound on h that is necessary for equilibrium solutions to exist. We observe that if we have exact equality in (4) then (3) reduces to $P = K/2$.

We now suppose that (4) is satisfied and address the question of whether the equilibrium solutions given by (3) are nonnegative. (In applications, negative solutions are not considered physically relevant.) The solution

$$P = \frac{K}{2} + \frac{K}{2}\sqrt{1 + \frac{4h}{Kr}} \tag{5}$$

is always greater than zero, so it suffices to consider the solution

$$P = \frac{K}{2} - \frac{K}{2}\sqrt{1 + \frac{4h}{Kr}}, \tag{6}$$

which is nonnegative when

$$\sqrt{1 + \frac{4h}{Kr}} \leq 1.$$

This condition simplifies to $h \leq 0$. Notice that the solution (6) is identically zero when $h = 0$ and is positive when $h < 0$.

3 Analysis

The results above show that not every choice of parameters r, K, h lead to models that admit equilibrium solutions. Rather, the condition (4) must be satisfied in order for an equilibrium solution to exist. One may interpret this condition as stating that the forcing rate h must not be more negative than the universal rate $-Kr/4$ that is determined by the capacity of the habitat and the ideal growth rate of the population. In particular, the carrying capacity K and ideal growth rate r determine a maximal rate at which a individuals may be removed from the population while still remaining in an equilibrium state. In the case that the forcing is at this maximally negative level, then the only nonzero equilibrium state is precisely half the carrying capacity.

In the case that the forcing term h is negative but greater than the critical rate of $-Kr/4$, then there are two positive equilibrium solutions, both of which solutions are less than carrying capacity. A negative value of h may be interpreted as "harvesting" or emigration of the population away from the habitat described by the model. In such a situation, our analysis states that steady state populations in the presence of harvesting/emigration are less than the habitat itself is predicted to sustain.

If the forcing term is zero, then there are two equilibrium solutions, the well-known equilibria $P = 0$ and $P = K$ for the unmodified logistic model.

Finally, if the forcing term h is positive then there exists only one nonnegative equilibrium solution, which is strictly larger than K. Positive values of the forcing constant may be interpreted as immigration. Our analysis states that in the case of constant immigration, the size of a steady state population will be greater than the population that the habitat itself is assumed to be able to sustainably support.

4 Conclusion

We have examined a logistic population model with a constant forcing term. In the case that the additional term is positive, we find that there exists only one positive equilibrium solution; this equilibrium population is larger than the carrying capacity of the habitat.

However, if the harvesting/hatchery term is negative then the picture is more complicated. From the ideal growth rate and the carrying capacity we may compute a "critical" growth rate. If the additional term is negative, but greater than this critical rate, then there are two positive equilibrium solutions, both of which are less than the carrying capacity. If the forcing term is exactly equal to this critical rate then there is only one equilibrium solution—this solution is half the carrying capacity.

Finally, if the forcing term is less than the critical rate then there are no equilibrium solutions. Thus the carrying capacity and ideal growth rate determine a harvesting threshold below which no equilibrium state is possible.