4.3 Superposition Principles

We now make use of vectors and matrices to study systems of linear systems of equations. Throughout, it is important to be able to translate seamlessly between writing systems as two independent equations and in vector-matrix form:

$$\begin{aligned}\frac{dy_1}{dt} &= ay_1 + by_2 \\ \frac{dy_2}{dt} &= cy_1 + dy_2\end{aligned} \qquad \leftrightarrow \qquad \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.3.1}$$

Activity 4.3.1. *Write the system*

$$\frac{dy_1}{dt} = y_2 \qquad \frac{dy_2}{dt} = y_1 + y_2$$

in vector-matrix notation.

We now consider the problem of constructing solutions to linear systems of equations. In order to motivate our approach, consider the following activity.

Activity 4.3.2. *Consider the equation*

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

1. *Show that* $Y_1(t) = e^{5t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is a solution.*
2. *Show that* $Y_2(t) = e^{2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ *is a solution.*
3. *Show that* $Y(t) = 3Y_1(t) + 7Y_2(t)$ *is a solution.*
4. *Explain why* $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$.
5. *Show that* $Y_1(0)$ *and* $Y_2(0)$ *are independent. Find* α *and* β *such that*

$$\alpha Y_1(0) + \beta Y_2(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

6. *With* α *and* β *as in the previous part, what initial value problem does* $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$ *solve?*

Through Activity 4.3.2 we learn the following important lesson:

Theorem (The Superposition Principle)**.** *Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions to a linear equation. Then any linear combination $\alpha Y_1(t) + \beta Y_2(t)$ is also a solution to the linear equation.*

As we saw in Activity 4.3.2, the superposition principle allows us to formulate a recipe for solving the initial value problem

$$\frac{d}{dt}Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad Y(0) = \begin{pmatrix} \heartsuit \\ \bigstar \end{pmatrix}. \tag{4.3.2}$$

The recipe is as follows:

1. Find two independent solutions $Y_1(t)$ and $Y_2(t)$ to the differential equation.
2. Choose constants α and β so that
$$\alpha Y_1(0) + \beta Y_2(0) = \begin{pmatrix} \heartsuit \\ \bigstar \end{pmatrix}.$$
3. Then the function
$$Y(t) = \alpha Y_1(t) + \beta Y_2(t)$$
is the unique solution to the linear IVP (4.3.2).

We make some comments on this procedure:

1. The recipe essentially puts all the work in to the first step. At this stage we don't know any systematic methods for actually constructing any solutions to the equation at all. So we still have our work cut out for us – we need to figure out how to get our hands on these two independent solutions. That's precisely the work we turn to in the next section.

   Even though the first step leaves us with work to do, the recipe is still very useful: Finding two solutions is a much simpler task than finding all solutions.

2. If we don't fix the constants α and β, then the linear combination
$$Y(t) = \alpha Y_1(t) + \beta Y_2(t)$$
is called a ***general solution*** to the differential equation. Once we have a general solution, then we can easily address the initial value problem by choosing the constants in order to satisfy the initial condition.

Activity 4.3.3. *Consider the initial value problem*

$$\frac{d}{dt}Y = \begin{pmatrix} 2 & 3 \\ 7 & 6 \end{pmatrix} Y \qquad Y(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

1. *Verify that*

$$Y_1(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{9t}\begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

   are solutions to the differential equation.

2. *Choose α and β so that*

$$\alpha Y_1(0) + \beta Y_2(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

3. *Construct the solution to the IVP.*

Exercise 4.3.1. Verify that $Y_1(t) = e^{6t}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $Y_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are both solutions of the system equations

$$\frac{d}{dt}Y = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} Y.$$

Exercise 4.3.2. Consider the system of equations

$$\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

1. Verify that $Y_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Y_2(t) = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are both solutions of this system equations.

2. By the Superposition Principle you now know infinitely many solutions of the system. What are they?

3. Solve the IVP:

$$\frac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4.4 Eigenstuff & straight line solutions

In the previous section we discovered a recipe for constructing solutions to linear equations. In this section we address the first step of that procedure, finding two independent solutions Y_1 and Y_2 to our linear system.

To motivate our approach, let's recall the example appearing in Activity 4.3.2. In that case, we were presented with two solutions:

$$Y_1(t) = e^{5t}\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{2t}\begin{pmatrix}2\\-3\end{pmatrix}.$$

Notice that both of these solutions trace out straight lines in the phase plane; a plot of them is here:

[graphic needed]

This example motivates an approach to finding solutions: We look for ***straight line solutions*** of the form

$$Y(t) = s(t)\begin{pmatrix}x\\y\end{pmatrix}, \tag{4.4.1}$$

where $s(t)$ is some function and x, y are fixed constants. Clearly not all solutions to linear equations are straight line solutions. But at this stage we don't need to construct all solutions – all we need to do is find some pair of independent solutions. So it makes sense to go looking for solutions that take a simple form, such as traversing a straight line.

Suppose, therefore, that we are looking for solutions to the linear equation

$$\frac{d}{dt}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}. \tag{4.4.2}$$

If we plug in a generic straight line function of the form (4.4.1) we find that we need

$$s'(t)\begin{pmatrix}x\\y\end{pmatrix} = s(t)\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

We can assume that $s(t)$ is not the zero function (because otherwise we don't get an interesting solution) and rewrite this equation as

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{s'(t)}{s(t)}\begin{pmatrix}x\\y\end{pmatrix}. \tag{4.4.3}$$

We now make an interesting observation: The left side of (4.4.3) does not depend on t: the entries of the matrix are constant and the entries of the vector are constant. Since the left side is constant in time, the right side must be as well. Thus we conclude the following: If(4.4.1) is going to be a straight line solution to (4.4.2), then it must be the case that $s'(t)/s(t)$ is constant. It is traditional to label this constant with the Greek letter λ. Replacing $s'(t)/s(t)$ by λ, the condition (4.4.3) becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.4.4}$$

together with the condition that

$$\frac{s'(t)}{s(t)} = \lambda. \tag{4.4.5}$$

Notice something amazing: the equation (4.4.4) does not involve any calculus. . . or even any functions! It is simply an algebraic equation where there are three unknowns: x, y, and λ. In fact, (4.4.4) is really just two equations with three unknowns. So it seems reasonable that we would be able to find several solutions. In other words, the prospects of finding straight-line solutions to (4.4.2) seem rather good.

The discussion in the previous paragraph suggests an approach for finding straight line solutions: First, find

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \lambda$$

that satisfy (4.4.4). Second, using that value of λ, find a function $s(t)$ that satisfies (4.4.5). Finally, building our straight line solution using the formula (4.4.1) that we started with.

Before attempting to execute this procedure to specific examples, it is worth taking a moment to recall why we are seeking straight line solutions in the first place. Remember that our goal is to construct a general solution to (4.4.2) of the form

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t), \tag{4.4.6}$$

where $Y_1(t)$ and $Y_2(t)$ are independent solutions. Our plan is to find $Y_1(t)$ and $Y_2(t)$ that are straight line solutions of the form (4.4.1) by solving (4.4.4) and (4.4.5). Since in (4.4.6) we will be multiplying Y_1 and Y_2 by arbitrary constants α and β, we don't need to worry about including any free constants in our scaling solution. This implies that we can take the solution to (4.4.5) to be simply

$$s(t) = e^{\lambda t}$$

and that all the difficult work involves finding solutions to (4.4.4). Furthermore, since we want $Y_1(t)$ and $Y_2(t)$ to be independent, we do not want both x and y in (4.4.4) to be zero.

We now turn to the problem of finding x, y, λ satisfying (4.4.4), where we require that we don't have both x and y zero. This problem is called the ***eigenvalue problem*** for the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.4.7}$$

We write equation (4.4.4) as

$$\begin{matrix} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{matrix} \quad \leftrightarrow \quad \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.4.8}$$

We now take a moment to study equations of this form; for convenience, we set $A = a - \lambda$, $B = b$, $C = c$, $D = (d - \lambda)$ so that the system is

$$\begin{matrix} Ax + By = 0 \\ Cx + Dy = 0. \end{matrix} \quad \leftrightarrow \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.4.9}$$

By multiplying the first equation by D and the second equation by B, and then subtracting, we find that

$$(AD - BC)x = 0.$$

Similarly by multiplying the first by C and the second by A, and then subtracting, we find that

$$(AD - BC)y = 0.$$

Thus we see that in order to have x and y satisfying (4.4.9) and have at least one of x or y be not zero, then we must have

$$AD - BC = 0.$$

The quantity $AD - BC$ is called the ***determinant*** of the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

What we have just discovered is that in order to have a non-zero solution to (4.4.9) the determinant of the matrix must be zero.

We now apply this knowledge to (4.4.8). In order to have one of x or y not equal to zero, we must have

$$(a-\lambda)(d-\lambda) - bc = 0. \tag{4.4.10}$$

The equation (4.4.10) is called the ***characteristic equation*** for the matrix (4.4.7).

The solutions λ to the characteristic equation are called the ***eigenvalues*** of the matrix. The eigenvalues are precisely the values of λ for which we can find solutions x, y to (4.4.4) where at least one of x, y is not zero. Thus in order to construct our straight line solutions, we proceed as follows:

1. Find the eigenvalues λ of the matrix appearing in our linear equation.

2. For each eigenvalue λ, find a corresponding non-zero vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

   satisfying (4.4.4). This vector is called an ***eigenvector*** associated to eigenvalue λ.

3. For each eigenvalue, let $s(t) = e^{\lambda t}$. Use this function, together with the associated eigenvector, to construct the straight line solution as in (4.4.1).

Assuming that this procedure yields two independent straight line solutions, then we are able to construct a general solution to the differential equation using the superposition principle.

Example 4.4.1. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

The characteristic equation for the matrix is

$$(2-\lambda)^2 - 1 = 0.$$

The solutions to the characteristic equation are

$$\lambda_1 = 3 \quad \textit{and} \quad \lambda_2 = 1.$$

We focus on $\lambda_1 = 3$. *In this case* (4.4.4) *becomes*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \leftrightarrow \qquad \begin{aligned} 2x + y &= 3x \\ x + 2y &= 3y \end{aligned}$$

Both of these equations reduce to $x = y$. *Thus we choose our eigenvector to be*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $s_1(t) = e^{\lambda_1 t} = e^{3t}$, *the corresponding straight line solution is*

$$Y_1(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now focus on $\lambda_1 = 1$. *In this case* (4.4.4) *becomes*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \leftrightarrow \qquad \begin{aligned} 2x + y &= x \\ x + 2y &= y \end{aligned}$$

Both of these equations reduce to $x = -y$. *Thus we choose our eigenvector to be*

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since $s_2(t) = e^{\lambda_2 t} = e^{t}$, *the corresponding straight line solution is*

$$Y_2(t) = e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since the two functions s_1 *and* s_2 *do not simultaneously vanish, and since the two eigenvectors are independent, we have have constructed two independent straight line solutions. Using the superposition principle, we conclude that a general solution to the differential equation is*

$$Y(t) = \alpha e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Activity 4.4.1. *Find the general solution to the system*

$$\frac{d}{dt} Y = \begin{pmatrix} 4 & 7 \\ 5 & 6 \end{pmatrix} Y$$

Activity 4.4.2. *Find the general solution to the system*

$$\frac{d}{dt}Y = \begin{pmatrix} 9 & 4 \\ 5 & 1 \end{pmatrix} Y$$

We have now developed an approach for finding solutions to linear systems of equations. Our plan now is to use this approach in order to understand the phase space diagrams for these systems. In particular, we want to use the method of "eigenstuff" to understand the stability of the equilibrium point at $(y_1, y_2) = (0, 0)$. As you will soon see, the stability of this equilibrium is determined primarily by the eigenvalues of the matrix that determines the equation. These eigenvalues, of course, are the solutions to the characteristic equation (4.4.10). Since the characteristic equation is quadratic, we see that there are a number of cases to consider: the case of two real solutions, the case of one real solution, and the case of no real solutions. In the next several sections we systematically deal with each of these cases, beginning with the case when there are two distinct real eigenvalues.

Exercise 4.4.1. Find the eigenvalues and the corresponding eigenvectors of the following matrices:

1. $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$
2. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
3. $\begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}$

Exercise 4.4.2. Find the explicit solution of the following IVP.

$$\frac{dY}{dt} = \begin{pmatrix} 11 & 30 \\ -4 & -11 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} 8 \\ -3 \end{pmatrix}.$$

4.5 Linear theory: Distinct real eigenvalues

In this section we study linear equations

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

in the case that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has two distinct, real eigenvalues $\lambda_1 < \lambda_2$. There are three general situations, and one special situation, that we consider:

- the positive case, when $0 < \lambda_1 < \lambda_2$;

- the negative case, when $\lambda_1 < \lambda_2 < 0$;
- the mixed case, when $\lambda_1 < 0 < \lambda_2$; and
- the zero case, when either $\lambda_1 = 0$ or $\lambda_2 = 0$.

Positive case

In the case that both eigenvalues are positive, both straight line solutions

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix}\heartsuit\\ \spadesuit\end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{\lambda_2 t}\begin{pmatrix}\diamondsuit\\ \clubsuit\end{pmatrix}$$

are moving away from the equilibrium at $(y_1, y_2) = (0, 0)$ as t increases. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t).$$

moves away from the equilibrium as well. In this situation we say that the equilibrium $(0, 0)$ is a ***source***; source equilibria are unstable.

We are assuming that $0 < \lambda_1 < \lambda_2$. This means when $t \gg 0$ we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$. Thus as $t \to \infty$ the solution $Y_2(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$. Thus as $t \to -\infty$, the solution $Y_1(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_1(t)$.

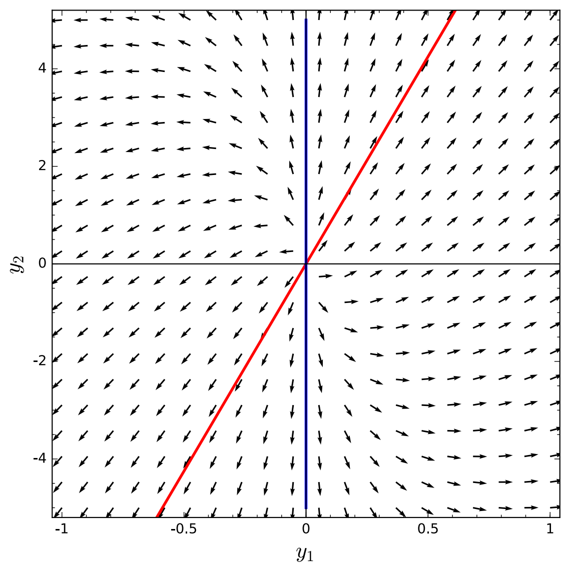
Example 4.5.1. *Consider the equation*

$$\frac{d}{dt}Y = \begin{pmatrix}5 & 0\\ 17 & 3\end{pmatrix} Y.$$

We compute the eigenvalues to be $\lambda_1 = 3$ and $\lambda_2 = 5$. The corresponding straight line solutions are

$$Y_1(t) = e^{3t}\begin{pmatrix}0\\ 1\end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{5t}\begin{pmatrix}2\\ 17\end{pmatrix}.$$

The phase diagram for this system is:



Here $Y_1(t)$ (and its negative) appears in blue, while $Y_2(t)$ (and its negative) appears in red. Notice that as $t \to \infty$, typical solutions move parallel to $Y_2(t)$, while as $t \to -\infty$ solutions move parallel to Y_1.

Negative case

In the case that both eigenvalues are negative, both straight line solutions

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix}\heartsuit\\ \spadesuit\end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{\lambda_2 t}\begin{pmatrix}\diamondsuit\\ \clubsuit\end{pmatrix}$$

are moving towards the equilibrium at $(y_1, y_2) = (0,0)$ as t increases. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t).$$

moves towards the equilibrium as well. In this situation we say that the equilibrium $(0,0)$ is a ***sink***; source equilibria are stable.

We are assuming that $\lambda_1 < \lambda_2 < 0$. This means when $t \gg 0$ we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$. Thus as $t \to \infty$ the solution $Y_2(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$. Thus as $t \to -\infty$, the solution $Y_1(t)$ dominates and any general solution $Y(t)$ is moving parallel to $Y_1(t)$.

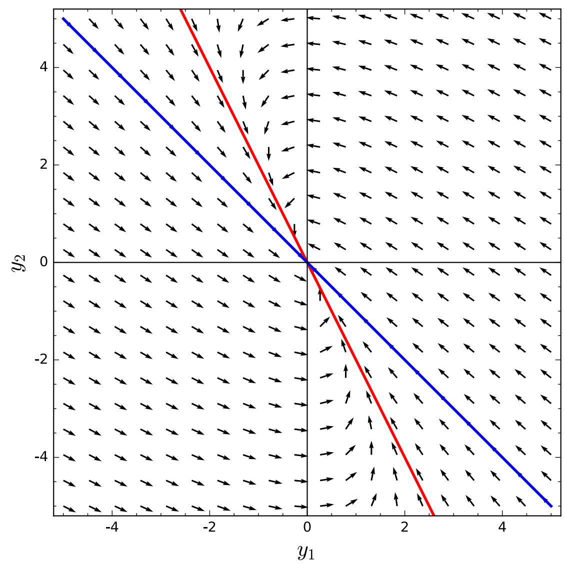
Example 4.5.2. *Consider the example*

$$\frac{d}{dt}Y = \begin{pmatrix} -6 & -2 \\ 4 & 0 \end{pmatrix} Y.$$

We compute the straight line solutions to be

$$Y_1(t) = e^{-4t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{-2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The phase diagram for this system is:



Here $Y_1(t)$ (and its negative) appears in blue, while $Y_2(t)$ (and its negative) appears in red. Notice that as $t \to \infty$, typical solutions move parallel to $Y_2(t)$, while as $t \to -\infty$ solutions move parallel to Y_1.

Mixed case

We now consider the case where $\lambda_1 < 0 < \lambda_2$. In this situation, the straight line solution

$$Y_1(t) = e^{\lambda_1 t}\begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix}$$

is moving towards the equilibrium at $(y_1, y_2) = (0, 0)$ as t increases, while the straight line solution

$$Y_2(t) = e^{\lambda_2 t}\begin{pmatrix} \diamondsuit \\ \clubsuit \end{pmatrix}$$

is moving away from the equilibrium. Consequently, the general solution

$$Y(t) = \alpha Y_1(t) + \beta Y_2(t)$$

ultimately moves away from the equilibrium point. In this situation we way that the equilibrium point $(0, 0)$ is a ***saddle***; saddle equilibria are unstable.

Notice that when $t \gg 0$ we have $e^{\lambda_1 t} \approx 0$. Thus as $t \to \infty$, the solution $Y_1(t)$ approaches zero and any general solution $Y(t)$ approaches $Y_2(t)$.

Similarly, when $t \ll 0$ we have $e^{\lambda_2 t} \approx 0$. Thus as $t \to -\infty$, the solution $Y_2(t)$ approaches zero and any general solution $Y(t)$ approaches $Y_1(t)$.

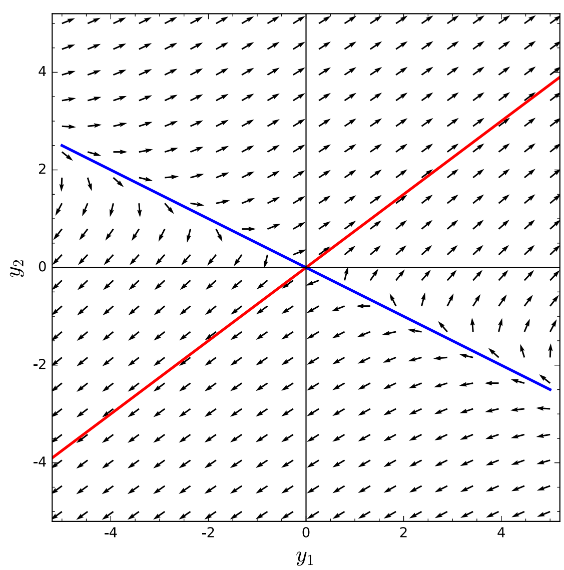
Example 4.5.3. *Consider the equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 3 & 8 \\ 3 & 5 \end{pmatrix} Y.$$

We compute the straight line solutions to be

$$Y_1(t) = e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad Y_2(t) = e^{9t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

The phase diagram for the system is:



Notice that as $t \to \infty$ all solutions approach either $Y_2(t)$ or its negative, which appear in red. As $t \to -\infty$, solutions all approach either $Y_1(t)$ or its negative, which appear in blue.

Zero case

There are actually two different possible ways one eigenvalue could be zero:

$$\lambda_1 < \lambda_2 = 0 \qquad \text{or} \qquad 0 = \lambda_1 < \lambda_2.$$

We treat the second case, and leave the first case as an exercise.

Suppose, then that $\lambda_1 = 0$ and $\lambda_2 > 0$. This means that the second straight line solution takes the form

$$Y_2(t) = e^{\lambda_2 t} \begin{pmatrix} \heartsuit \\ \spadesuit \end{pmatrix},$$

while the second straight line solution takes the form

$$Y_1(t) = \begin{pmatrix} \diamondsuit \\ \clubsuit \end{pmatrix}.$$

Notice that $Y_1(t)$ is actually an equilibrium solution! Since any multiple of $Y_1(t)$ is also a solution, we see that there are a whole line of equilibrium solutions in the phase plane.

Since $\lambda_2 > 0$, we have $e^{\lambda_2 t} \to \infty$ as $t \to \infty$. This means that as $t \to \infty$, general solutions move away from the line of equilibrium solutions as $t \to \infty$.

An equilibrium point that lies on a line of other equilibrium points is called a ***center manifold point***. The study of such points is, unfortunately, beyond the scope of this course. Thus for our purposes here, we simply "do not comment" on the stability of the equilibrium point $(y_1, y_2) = (0, 0)$ in this case.

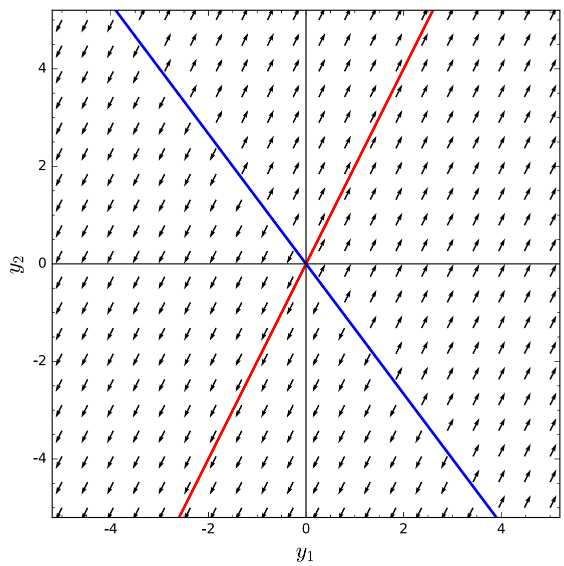
Example 4.5.4. *Consider the equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} Y.$$

We compute that the straight line solutions are

$$Y_1(t) = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \qquad \textit{and} \qquad Y_2(t) = e^{10t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The phase diagram for this equation is the following:



Notice the line of equilibrium solutions (in blue). Furthermore, all solutions not on that line are moving parallel to the non-constant solution $Y_2(t)$.

Exercise 4.5.1. Study each of the following "systems" by addressing the following questions:

- Find the general solution of the system;
- Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.
- Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition $Y(0)$.
- Discuss the stability of the equilibrium solution $Y(t) = 0$.

1. $\dfrac{dY}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} Y$
2. $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ 3 & 0 \end{pmatrix} Y;$
3. $\dfrac{dY}{dt} = \begin{pmatrix} 0.41 & 0.12 \\ 0.12 & 0.34 \end{pmatrix} Y;$
4. $\dfrac{dY}{dt} = \begin{pmatrix} 4 & 15 \\ -2 & -7 \end{pmatrix} Y;$

Exercise 4.5.2. A market researcher established that the daily profits of two competing stores, *Ethel's Knick-knack Heaven* and *Irma's Antiques*, relate to each other as in the system:

$$\frac{dx}{dt} = 6x - 2y$$
$$\frac{dy}{dt} = -2x + 3y$$

here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

1. Draw the phase portrait of this system. Use the least amount of computations possible.

2. Use the phase portrait only to discuss the long-term fate of these stores if their current profits are:

   (a) $x_0 = \$90$ and $y_0 = \$240$

   (b) $x_0 = \$100$ and $y_0 = \$120$

   (c) $x_0 = \$95$ and $y_0 = \$180$

3. Use the phase portrait to find the relationship between the current profits which would allow both stores to stay in business.

4. Find (analytically) the particular solution of the model corresponding to the current profits of $x_0 = \$90$ and $y_0 = \$240$. Use the knowledge of this particular solution to determine when Ethel's profits are going to be biggest. How big is this profit?

5. Do the same for Irma's store and the current profits of $x_0 = \$95$ and $y_0 = \$180$.

Exercise 4.5.3. Two 100 gallon mixing tanks TANK_1 and TANK_2 are connected to each other with two pipes, PIPE_1 and PIPE_2. The tanks are both completely filled with salty water. The salty water from TANK_1 flows through PIPE_1 to TANK_2 at the (continuous) rate of 8 gal/hr. The salty water from TANK_2 flows through PIPE_2 to TANK_1 at the (continuous) rate of 2 gal/hr. The volume of TANK_1 is kept constant by continuous adding of pure water at the rate of 6 gallons per hour. Likewise, the volume of TANK_2 is kept constant by continuous draining at the rate of 6 gallons per hour. Everything is always kept 'perfectly well mixed.'

1. Let $s_1(t)$ and $s_2(t)$ be the amount of salt (in pounds, say) in TANK_1 and TANK_2 (respectively). Write a linear system of differential equations modeling $s_1(t)$ and $s_2(t)$.

2. Draw the phase portrait of this system.

3. What can you say about “the fate of” s_1 and s_2? Is it true that

$$\lim_{t\to+\infty} s_1(t) = \lim_{t\to+\infty} s_2(t) = 0 \quad ?$$

   Which of the two tanks will have more salt in the long run? Does your answer depend on how much salt the tanks initially had?

4.6 Complex numbers

Suppose we want to find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}. \tag{4.6.1}$$

The characteristic equation for this matrix is

$$(1-\lambda)(3-\lambda)+5=0,$$

which we rewrite as

$$\lambda^2 - 4\lambda + 8 = 0.$$

The quadratic formula tells us that the solutions are

$$\lambda_+ = \frac{4+\sqrt{-16}}{2} \qquad \text{and} \qquad \lambda_- = \frac{4-\sqrt{-16}}{2}$$

Unfortunately, these are not real numbers. Thus the linear differential equation defined by the matrix (4.6.1) does not have straight line solution. However, this does not mean that all hope for using superposition to understand the differential equation is lost. Rather, we need to do a bit more work. That work involves using complex numbers, which is the subject of this section.

Complex numbers are specifically designed to ensure that all quadratic equations have solutions. This is accomplished by the inclusion of a new number i defined by $i^2 = -1$. A ***complex number*** is defined to be a number of the form

$$a + bi,$$

where both a and b are real numbers. The number a is called the ***real part*** of $a+bi$ and the number b is called the ***imaginary part*** of $a+bi$. Note that both the real part and the imaginary part are themselves real numbers.

It is common to use the following notation for the real and imaginary parts of a complex number:

$$\mathrm{Re}(a+bi) = a \qquad \text{and} \qquad \mathrm{Im}(a+bi) = b.$$

Example 4.6.1. *Consider the equation*

$$\lambda^2 + 3\lambda + 4 = 0.$$

From the quadratic formula we see that solutions are

$$\begin{aligned} \lambda_+ &= \frac{-3+\sqrt{9-16}}{2} = -\frac{3}{2} + \frac{\sqrt{7}}{2}\, i \\ \lambda_- &= \frac{-3-\sqrt{9-16}}{2} = -\frac{3}{2} - \frac{\sqrt{7}}{2}\, i. \end{aligned}$$

Thus

$$\mathrm{Re}(\lambda_+) = -\frac{3}{2} \qquad \textit{and} \qquad \mathrm{Im}(\lambda_+) = \frac{\sqrt{7}}{2}.$$

Complex numbers are a very interesting and important part of mathematics. I strongly encourage all of you to take our Complex Variables course to learn more. For now, however, I will present only those few results that we need for this course.

First, note that we can perform all of the "usual" algebraic procedures on complex numbers. Adding/subtracting and multiplying/dividing is rather straightforward, provided we remember that $i^2 = -1$. For example,

$$\begin{gathered} (2-3i)+(7+2i) = 9-i, \\ (2-3i)(7+2i) = 14-17i-6i^2 = 20-17i. \end{gathered}$$

One can also define what it means to compute the exponential function of a complex number by means of the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots.$$

In particular, we have

$$\begin{aligned} e^{a+ib} &= e^a e^{ib} \\ &= e^a \left[1 + (ib) + \frac{1}{2}(ib)^2 + \frac{1}{6}(ib)^3 + \frac{1}{24}(ib)^4 + \dots\right] \\ &= e^a \left[\left(1 - \frac{1}{2}b^2 + \frac{1}{24}b^4 - \dots\right) + i\left(b - \frac{1}{6}b^3 + \dots\right)\right] \\ &= e^a \left(\cos(b) + i\sin(b)\right). \end{aligned}$$

In the case that $a = 0$ this identity becomes

$$e^{ib} = \cos(b) + i\sin(b)$$

and is known as ***Euler's formula***.

Euler's formula has several interesting and useful applications.

Activity 4.6.1. *Use Euler's formula to deduce that*

$$e^{-ib} = \cos(b) - i\sin(b).$$

Conclude from this that

$$\cos(b) = \frac{e^{ib} + e^{-ib}}{2} \qquad \textit{and} \qquad \sin(b) = \frac{e^{ib} - e^{-ib}}{2}.$$

Activity 4.6.2. *Since $e^{2\theta} = (e^{\theta})^2$, Euler's formula implies that*

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2.$$

Multiply out the right side of this identity in order to deduce the double angle formulas.

Activity 4.6.3. *Use the fact that $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$, together with Euler's formula, to deduce the angle sum formulas.*

Finally, we return to the problem that motivated this discussion: the eigenvalues of the matrix (4.6.1). The computation above shows that the eigenvalues are

$$\lambda_+ = 2 + 2i \qquad \text{and} \qquad \lambda_- = 2 - 2i.$$

We can subsequently compute the eigenvectors to be

$$\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

In the next section we discuss how to use complex eigenvalues and eigenvectors in order to obtain real solutions to linear systems of equations.

Exercise 4.6.1. Put the following expressions into the form $a + b\,i$, where a,b are real numbers.

1. $(1 + 2i)(3 - 4i)$

2. $\dfrac{1}{2+i}+\dfrac{1}{1-2i}$

3. $\dfrac{2+3i}{1+i}$

Exercise 4.6.2. Solve the quadratic equation $x^2+x+1=0$ within the set of complex numbers.

Exercise 4.6.3. Solve the system of equations:

$$\begin{aligned} x+iy &= 2 \\ 2ix-y &= 3i. \end{aligned}$$

Exercise 4.6.4. Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} -2 & -9 \\ 1 & -2 \end{pmatrix}.$$

Exercise 4.6.5. Derive the Euler formula using Taylor expansions.

Exercise 4.6.6. Decompose into real and imaginary parts.

1. $e^{\frac{i\pi}{4}}$
2. $e^{-1+\pi i}$
3. $2e^{1+i}+2e^{1-i}$
4. $e^{-(2+\pi i)t}$
5. $e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} - e^{-it}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$
6. $e^{(1+i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + e^{(1-i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

Exercise 4.6.7. Assume that a, b and t are some real numbers with $b \neq 0$.

1. Identify the real and the imaginary part of the expression $e^{(a+ib)t}$.
2. Treat the real and the imaginary part as individual functions of the independent t variable: $f(t)$ and $g(t)$. Graph these two functions. Note that the graphs will look radically different depending on whether a is positive, zero, or negative. Explore all three situations.

4.7 Linear theory: Complex eigenvalues

In this section we explore how to use complex eigenvalues and eigenvectors to construct real solutions to linear systems of equations. We begin with an example.

Example 4.7.1. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix} Y.$$

In the previous section we found that there were two eigenvalues

$$\lambda_+ = 2 + 2i \qquad \text{and} \qquad \lambda_- = 2 - 2i$$

having the associated eigenvectors

$$\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

These give rise to complex solutions to the differential equation

$$Y_+(t) = e^{(2+2i)t}\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \qquad \text{and} \qquad Y_-(t) = e^{(2-2i)t}\begin{pmatrix} 1 \\ 1-2i \end{pmatrix}.$$

A general complex solution to the equation is therefore

$$\begin{aligned} Y(t) &= Ae^{(2+2i)t}\begin{pmatrix} 1 \\ 1+2i \end{pmatrix} + Be^{(2-2i)t}\begin{pmatrix} 1 \\ 1-2i \end{pmatrix} \\ &= Ae^{2t}\left\{\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}\right\} \\ &\quad + Be^{2t}\left\{\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} - i\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}\right\}. \end{aligned}$$

Notice that the real parts of $Y_+(t)$ and $Y_-(t)$ are the same, while the imaginary parts are opposites. We can use this to our advantage in order to construct real solutions.

First, choose $A = \frac{1}{2}$ and $B = \frac{1}{2}$. In this case we obtain the solution

$$Y_1(t) = e^{2t}\begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix}.$$

Second, choose $A = \frac{1}{2i}$ and $B = -\frac{1}{2i}$. In this case we obtain the solution

$$Y_2(t) = e^{2t}\begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}.$$

These two solutions are independent of one another, and are real-valued functions! Thus we have succeeded in constructing two real, independent solutions. Consequently, we can form a (real) general solution to the differential equation:

$$Y(t) = \alpha e^{2t} \begin{pmatrix} \cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + \beta e^{2t} \begin{pmatrix} \sin(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}.$$

We also want to understand what the phase portrait for this equation looks like. We do this in two steps. First we analyze the solution $Y_1(t)$. Then we show that $Y_2(t)$ behaves in a qualitatively similar manner.

To understand how $Y_1(t)$ behaves we write it as

$$Y_1(t) = e^{2t}\cos(2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\sin(2t)\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Let's slowly build this function by modifying simpler functions.

1. *The function*

   $$\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   traverses the unit circle in the anti-clockwise direction; we can interpret this as alternating between having a position that is displaced from the origin in the

   $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   directions, with motion from the first vector to the second vector.

2. *We can modify this unit circle trajectory by replacing the vectors*

   $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   with other vectors. Thus the trajectory

   $$\cos(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

   alternates between displacement in the

   $$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

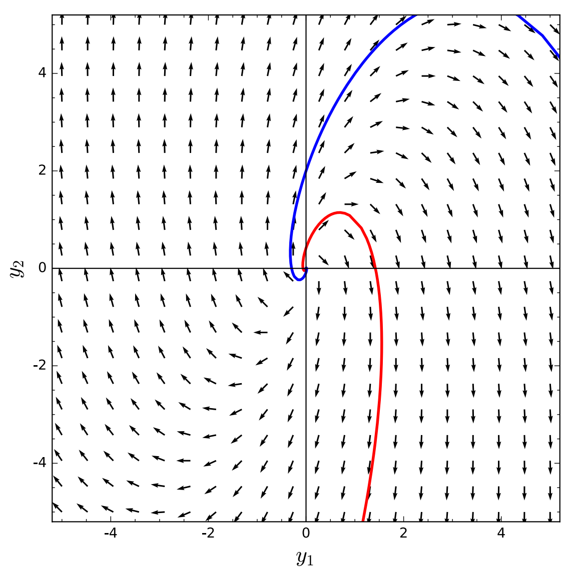
   directions, with motion from the first to the second. The result is an elliptical path, traversed in the clockwise direction.

3. *If we replace* $\cos(t)$ *and* $\sin(t)$ *by* $\cos(2t)$ *and* $\sin(2t)$*, the result is a trajectory that traverses the same path, but twice as fast.*

4. *Finally, to obtain* $Y_1(t)$ *we multiply the whole thing by* e^{2t}*. This has the effect of increasing the overall location outwards as* t *increases, changing the elliptical trajectory to an outwards spiral trajectory.*

Now that we have analyzed $Y_1(t)$ *we examine* $Y_2(t)$*. While we could repeat the process used above to examine* $Y_1(t)$*, it is more efficient to simply note that* $Y_2(t) = e^{-\pi/2} Y_1(t - \pi/4)$*. Thus the two follow similar trajectories, but are out of phase of one another.*

The following shows the phase diagram for the differential equation, with $Y_1(t)$ *in red and* $Y_2(t)$ *in blue.*



We can learn several lessons from Example 4.7.1 that help us to more efficiently analyze differential equations in the case of complex eigenvalues.

The first is that complex eigenvalues, and the corresponding complex solutions, come in pairs where the real parts are the same and the imaginary parts are opposites. In particular, we always get two solutions such that

$$Y_+(t) = \mathrm{Re}(Y_+(t)) + i\,\mathrm{Im}(Y_+(t))$$
$$Y_-(t) = \mathrm{Re}(Y_+(t)) - i\,\mathrm{Im}(Y_+(t)).$$

Using the superposition principle, we know that

$$Y_1(t) = \mathrm{Re}(Y_+(t)) = \frac{1}{2}Y_+(t) + \frac{1}{2}Y_-(t)$$
$$Y_2(t) = \mathrm{Im}(Y_+(t)) = \frac{1}{2i}Y_+(t) + \frac{1}{2i}Y_-(t)$$

are solutions. Thus in order to find two real, independent solutions all we need to do is to find $Y_+(t)$ and then take, separately, the real and imaginary parts.

The second lesson that we learn from the example above is that if $\lambda = a + ib$ then the real solution $Y_1(t)$ takes the form

$$Y_1(t) = e^{at}\cos(bt)\begin{pmatrix}\heartsuit\\ \spadesuit\end{pmatrix} + e^{at}\sin(bt)\begin{pmatrix}\diamondsuit\\ \clubsuit\end{pmatrix}$$

for some constants $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$. Thus the solution $Y_1(t)$ will oscillate between being displaced in two different directions, rotating in either a clockwise or anticlockwise direction. Furthermore,

- If $\mathrm{Re}(\lambda) > 0$ then solutions spiral outwards. In this case, the equilibrium at $(0, 0)$ is called a ***spiral source***. Spiral sources are unstable.

- If $\mathrm{Re}(\lambda) = 0$ then solutions traverse an elliptical trajectory. In this case, the equilibrium $(0, 0)$ is called a ***center***. Centers are generally considered to be unstable, but interpretations vary.

- If $\mathrm{Re}(\lambda) < 0$ then solutions spiral inward towards $(0, 0)$. In this case, the equilibrium at $(0, 0)$ is called a ***spiral sink***. Spiral sinks are stable.

The third lesson we learn from Example 4.7.1 is that the two solutions Y_1 and Y_2 traverse trajectories that differ only by scaling and phase. Thus if we are only interested in obtaining a qualitative understanding, it is enough to study $Y_1(t)$.

Finally, note that there is an easy trick for seeing whether a spiral solution is rotating in the clockwise or anticlockwise direction. Consider the solution that passes through the point $(1, 0)$ at that point, the velocity vector is determined by the differential equation. If the vector is pointing in to the first quadrant, then the solution traverses anticlockwise; if the vector is pointing in to the fourth quadrant, then the solution is traversing in the clockwise direction.

Example 4.7.2. *Consider the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} Y.$$

We compute the eigenvalues to be $\lambda = \pm 3i$. *Already at this stage we know that solutions traverse elliptical paths around the center-type equilibrium at* $(0,0)$. *Furthermore, we know that the solution passing through the point* $(1,0)$ *has velocity vector*

$$\begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

at that point. Thus solutions traverse in the clockwise direction.

In order to obtain a general solution, we compute the eigenvector corresponding to $\lambda = 3i$, *which is*

$$\begin{pmatrix} 2 \\ -1+3i \end{pmatrix}.$$

Thus the first complex solution is

$$Y_+(t) = e^{3i} \begin{pmatrix} 2 \\ -1+3i \end{pmatrix} = \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + i \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

From this we may extract the real and imaginary parts, which we know are each real solutions:

$$Y_1(t) = \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} \quad \text{and} \quad Y_2(t) = \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}.$$

Consequently, the general solution is

$$Y(t) = \alpha \begin{pmatrix} 2\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} + \beta \begin{pmatrix} 2\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}$$

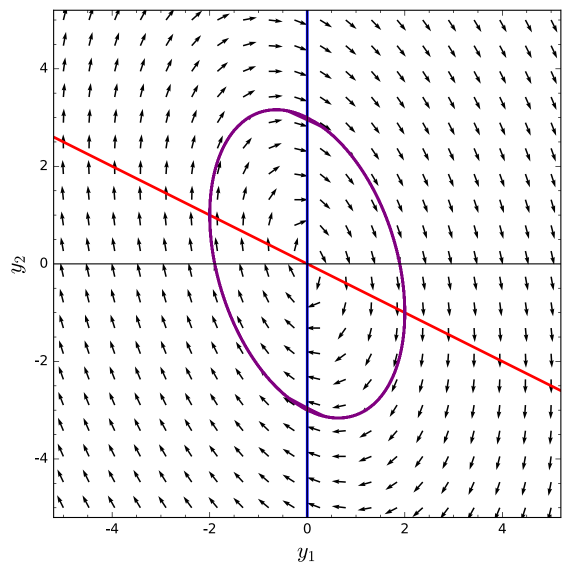
Finally, examining $Y_1(t)$ *we find*

$$Y_1(t) = \cos(3t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \sin(3t) \begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Thus we see that solutions oscillate between displacements in the directions

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -3 \end{pmatrix}. \tag{4.7.1}$$

The phase space diagram for this equation is as follows:



In the diagram, the trajectory of solution $Y_1(t)$ *is shown in purple. The two directions* (4.7.1) *appear in red and blue, respectively.*

Activity 4.7.1. *Analyze the differential equation*

$$\frac{d}{dt}Y = \begin{pmatrix} -8 & 13 \\ -2 & 2 \end{pmatrix} Y.$$

Determine the type of equilibrium point at $(0,0)$ *and the direction in which solutions traverse the phase plane. Use this to make a crude sketch of the phase diagram. Then find the general solution.*

Exercise 4.7.1.

1. Find the explicit solution of the following IVP.

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

2. Suppose $x_0 = 1$ and $y_0 = 2$. What is the corresponding solution to the IVP?

Exercise 4.7.2. Study each of the following "systems" by addressing the following questions:

- Find the general solution of the system;

- Draw the phase diagram for the system without any use of "technology". Then check your answer with Sage.
- Draw the solution curves in the t–x and the t–y planes.
- Discuss the long-term fate of the solutions of the system. Your answer potentially depends on the initial condition $Y(0)$.
- Discuss the stability of the equilibrium solution $Y(t) = 0$.

1. $\dfrac{dY}{dt} = \begin{pmatrix} 6 & 9 \\ -5 & -6 \end{pmatrix} Y;$

2. $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix} Y;$

3. $\dfrac{dY}{dt} = \begin{pmatrix} 5 & -3 \\ 6 & -1 \end{pmatrix} Y.$

Exercise 4.7.3. A market researcher established that the daily profits of two competing stores, *Ethel's Knick-knack Heaven* and *Irma's Antiques*, relate to each other as in the system:

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= -5x + 3y \end{aligned}$$

here $x(t)$ denotes the daily profit of Ethel's store and $y(t)$ denotes the daily profit of Irma's store. The time is measured in months.

1. Draw the phase portrait of this system. Use the least amount of computations possible.
2. Use the phase portrait only to discuss the long-term fate of these stores.

Exercise 4.7.4. Draw the phase diagrams of the following systems, using the least possible amount of computation.

1. $\dfrac{dY}{dt} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} Y$

2. $\dfrac{dY}{dt} = \begin{pmatrix} -5 & 2 \\ -3 & 0 \end{pmatrix} Y$

3. $\dfrac{dY}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} Y$

Exercise 4.7.5. The following problem concerns profits of two nearby stores; the model we shall use is based on a linear system of equations. The first store, the profit of which at time t we label by $x(t)$, was successful on its own until recently when a new store opened nearby. This second store, the profit of which at time t we label by $y(t)$, offers a lot of low-quality cheap merchandise. If it wasn't for the customers of the old store dropping by periodically the second store would not be able to survive.

The model applicable to these two stores is:

$$\begin{cases} \dfrac{dx}{dt} = x - 5y \\[2ex] \dfrac{dy}{dt} = 2x - y. \end{cases}$$

Currently, the "profits" of both stores are negative. Determine if the stores will ever recover, and what their long term fate is. Your supporting evidence should at least include a phase diagram.