2.2 Propagator functions

In this section we consider differential equations of the form

$$\frac{dy}{dt} = ry + f, \tag{2.2.1}$$

where both r and f are functions of t. We interpret (2.2.1) as a growth model with time-dependent relative growth rate r and forcing term f.

Example 2.2.1. *Suppose an investment is growing with variable annual interest rate $r(t) = 0.02 + 0.15\cos(t)$. Suppose furthermore that additional funds are added to the principle investment according to the function $f(t) = e^{-3t}$, measured in thousands of dollars per year. Let $V(t)$ be the value of the investment fund, measured in thousands of dollars, at time t, measured in years. Then growth of V is described by*

$$\frac{dV}{dt} = (0.02 + 0.15\cos(t))V + e^{-3t}.$$

The differential equation (2.2.1) is not separable. Nevertheless, we are able to obtain reasonable formulas for solutions. To accomplish this, we first consider the initial value problem in the special case that there is no forcing, namely:

$$\frac{dy}{dt} = ry \quad y(t_0) = y_0. \tag{2.2.2}$$

The differential equation in (2.2.2) *is* separable, and thus can be addressed using the methods of the previous section; see Exercise 2.2.1.

However, we can also arrive at an understanding of the formula for the solution to (2.2.2) by the following line of reasoning. First, in the case that r is constant, the solution to (2.2.2) is

$$y(t) = y_0 \exp\left(r(t - t_0)\right).$$

We rewrite this as

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r\, d\tau\right). \tag{2.2.3}$$

Suppose now that r is in fact a function. In the case the function given by (2.2.3) is

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right). \tag{2.2.4}$$

We can directly compute, using the Fundamental Theorem of Calculus and the chain rule, that

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right)\right] \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right) \frac{d}{dt}\int_{t_0}^{t} r(\tau)\, d\tau \\ &= y_0 \exp\left(\int_{t_0}^{t} r(\tau)\, d\tau\right) r(t) \\ &= r(t)y(t). \end{aligned} \tag{2.2.5}$$

Furthemore

$$y(t_0) = y_0 \exp\left(\int_{t_0}^{t_0} r(\tau)\, d\tau\right) = y_0 \exp(0) = y_0.$$

Thus the function $y(t)$ determined by (2.2.4) is the solution to (2.2.2).

Notice the structure of the formula (2.2.4) – it simply consists of the initial value y_0 multiplied by some function. If we define the function $P(t, s)$ by

$$P(t, s) = \exp\left(\int_{s}^{t} r(\tau)\, d\tau\right), \tag{2.2.6}$$

then the solution to (2.2.2) is given by $y(t) = P(t, t_0)y_0$.

The function $P(t, s)$ given by the formula (2.2.6) is called the ***propagator function*** for the rate function $r(t)$. Multiplication by $P(t, s)$ corresponds to propagation according to the differential equation in (2.2.2), starting at time s and ending at time t.

Propagator functions have the following properties:

1. $P(s, s) = 1$,
2. $\frac{d}{dt}P(t, s) = r(t)P(t, s)$,
3. $P(t, s) = P(s, t)^{-1}$,
4. $P(t_2, t_1)P(t_1, t_0) = P(t_2, t_0)$.

The first two properties imply that $P(t, s)$ is the unique solution to the initial value problem

$$\frac{dy}{dt} = r(t)y \qquad y(s) = 1. \tag{2.2.7}$$

We can also interpret the first property as the statement that propagating for a time interval of zero size does not change the value of y. The third property states that propagation from time s to time t is the opposite (in the multiplicative sense) of propagation from time t to time s. The fourth property states that propagation from time t_0 to time t_1 and then from time t_1 to time t_2 is the same as propagation from time t_0 to time t_2.

We now show how to use propagator functions in order to find an expression for the solution to the forced equation (2.2.1) with initial condition $y(t_0) = y_0$. First we develop a formula using a intuitive, non-rigorous reasoning; then we verify that the formula solves the desired initial value problem. We begin by supposing that the value of y at time t will be the sum of two terms, one due to the influence of the initial condition and one due to the influence of the forcing. The term due to the initial condition we suppose to be $P(t, t_0)y_0$ (just as in the case when there is no forcing).

The forcing term can be viewed as continuously contributing an "infinitesimal additional amount" to the system. In particular, at a given time s, the quantity $f(s)\,ds$ represents the amount added to the system at that time. (Note that $f(s)\,ds$ has the same units as y.) In order to compute the impact that $f(s)\,ds$ has on the value of $y(t)$ we use the propagator function; the contribution is $P(t, s)f(s)\,ds$. Finally we "add up" all of these infinitesimal contributions, starting with those at time t_0 and ending with those at time t. The result is that we expect the contribution to the value of $y(t)$ due to forcing to be

$$\int_{t_0}^{t} P(t, s)f(s)\,ds$$

and thus we conjecture that the solution to (2.2.1) with initial condition $y(t_0) = y_0$ is

$$y(t) = P(t, t_0)y_0 + \int_{t_0}^{t} P(t, s)f(s)\,ds. \tag{2.2.8}$$

The formula (2.2.8) is called ***Duhamel's formula***.

Let's now verify that the function given by (2.2.8) does indeed solve both (2.2.1) and satisfy the initial condition $y(t_0) = y_0$. The initial condition is straightforward to see, so we focus on the differential equation. It is convenient to use the identity

$$P(t, s) = P(t, t_0)P(t_0, s)$$

in order to rewrite (2.2.8) as

$$y(t) = P(t, t_0)y_0 + P(t, t_0)\int_{t_0}^{t} P(t_0, s)f(s)\,ds.$$

We compute using the properties of propagator functions that

$$\begin{aligned}
y'(t) &= \frac{d}{dt}\left[P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right] \\
&= \frac{d}{dt}\left[P(t,t_0)\right]y_0 + \frac{d}{dt}\left[P(t,t_0)\right]\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)\frac{d}{dt}\int_{t_0}^{t} P(t_0,s)f(s)\,ds \\
&= r(t)P(t,t_0)y_0 + r(t)P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds + P(t,t_0)P(t_0,t)f(t) \\
&= r(t)\left(P(t,t_0)y_0 + P(t,t_0)\int_{t_0}^{t} P(t_0,s)f(s)\,ds\right) + (1)f(t) \\
&= r(t)y(t) + f(t).
\end{aligned}$$

Thus we see that the formula (2.2.8) does indeed satisfy the IVP.

One interesting and important feature of (2.2.8) is that we see explicitly that the solution $y(t)$ is the sum of two parts: the ***homogeneous solution***

$$P(t,t_0)y_0,$$

which due to the initial condition y_0, and the ***particular solution***

$$\int_{t_0}^{t} P(t,s)f(s)\,ds$$

which due to the forcing. Later in the course we see encounter this same phenomenon.

Example 2.2.2. *Consider the initial value problem*

$$\frac{dy}{dt} = t^2 y \qquad y(0) = 17.$$

(Note that this equation is separable; see Exercise **??***.) In this case we have* $r(t) = t^2$*,* $f(t) = 0$*,* $t_0 = 0$*, and* $y_0 = 17$*. We compute the propagator function to be*

$$P(t,s) = \exp\left(\int_s^t \tau^2\,d\tau\right) = \exp\left(\frac{1}{3}t^3 - \frac{1}{3}s^3\right). \tag{2.2.9}$$

Thus the solution is

$$\begin{aligned}
y(t) &= \exp\left(\frac{1}{3}t^3\right)y_0 + \int_0^t \exp\left(\frac{1}{3}t^3 - \frac{1}{3}s^3\right)(0)\,ds \\
&= e^{t^3/3}y_0.
\end{aligned}$$

Example 2.2.3. *Let's use propagators to solve the initial value problem*

$$\frac{dy}{dt} = \cos t - 7 \qquad y(0) = \sqrt{17}.$$

In this IVP the rate function is $r(t) = -7$. Thus the propagator is given by

$$P(t, s) = \exp\left(\int_s^t -7 d\tau\right) = e^{-7(t-s)}.$$

The initial value $y_0 = \sqrt{17}$ is specified at time $t_0 = 0$. Thus the homogeneous solution is

$$P(t, 0)y_0 = \sqrt{17}e^{-7t}.$$

The forcing function is $f(t) = \cos t$; hence the particular solution is

$$\int_0^t P(t, s)f(s)ds = \int_0^t e^{-7(t-s)} \cos(s)\, ds = \frac{7}{50} \cos(t) - \frac{7}{50} e^{(-7t)} + \frac{1}{50} \sin(t).$$

Thus the solution to the initial value problem is

$$y(t) = \sqrt{17}e^{-7t} + \frac{7}{50} \cos(t) - \frac{7}{50} e^{(-7t)} + \frac{1}{50} \sin(t).$$

Activity 2.2.1. *Use propagators to solve the initial value problem*

$$\frac{dy}{dt} = 1 - \frac{y}{1-t} \qquad y(0) = 31.$$

Exercise 2.2.1. We arrived at the formula (2.2.4) by "intuiting" what the solution "ought" to be. A more rigorous way to proceed would be to notice that the differential equation in (2.2.2) is separable. Use this fact to directly obtain (2.2.4) by the method of separation of variables.

Exercise 2.2.2. Verify (by direct computation) the properties of the propagator function $P(t, s)$:

1. $P(s, s) = 1$,
2. $\frac{d}{dt}P(t, s) = r(t)P(t, s)$,
3. $P(t, s) = P(s, t)^{-1}$,
4. $P(t_2, t_1)P(t_1, t_0) = P(t_2, t_0)$.

Exercise 2.2.3. Solve the following IVP using the propagator method:

$$\frac{dy}{dt} = 2y - t \qquad y(0) = 1.$$

Exercise 2.2.4. Solve the following IVP using the propagator method:

$$\frac{dy}{dt} = \frac{y}{1+2t} - 7 \qquad y(0) = 1.$$

Exercise 2.2.5. Suppose money is invested in a volatile market that has an annual growth rate of $r(t) = 0.01 + 0.05\cos 10t$, where t is measured in years.

1. Make a plot of $r(t)$ over a 10 year time period. How should one interpret this growth rate?
2. Suppose there is an initial investment of \$100. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?
3. Suppose instead that no money is initially invested, but that one continuously adds to an investment at a rate of \$10 per year. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?

Exercise 2.2.6. Solve the initial value problem from Exercise 1.2.4.

2.3 Integrating factors

The method of propagators introduced in the previous section is important in part because it can be generalized to many situations besides those considered in that section; see Section 3.9. However, for simple initial value problems of the form

$$\frac{dy}{dt} = ry + f \quad y(t_0) = y_0, \tag{2.3.1}$$

where both r and f are functions of t, there is a “trick” – called the ***method of integrating factors*** – that is sometimes useful.

The idea of the trick is reverse engineer the product rule. Recall that for two functions u and y says that

$$\frac{d}{dt}[uy] = u\frac{dy}{dt} + \frac{du}{dt}u.$$

In order to apply this to (2.3.1) we write the differential equation as

$$\frac{dy}{dt} - ry = f.$$

If we multiply this by some function u we obtain

$$u\frac{dy}{dt} + (-ru)y = uf. \tag{2.3.2}$$

The left side of this last equation would look like the derivative of uy if

$$\frac{du}{dt} = -ru. \tag{2.3.3}$$

Let's assume that we have chosen the function u so that it is a solution to (2.3.3). Then (2.3.2) becomes

$$\frac{d}{dt}[uy] = uf.$$

We can integrate this last equation from time t_0 to time t, which leads to

$$u(t)y(t) = u(t_0)y(t_0) + \int_{t_0}^{t} u(\tau)f(\tau)\,d\tau.$$

Dividing by $u(t)$ and using the initial condition $y(t_0) = y_0$ leads to a formula for the solution to (2.3.1), namely

$$y(t) = \frac{u(t_0)}{u(t)}y_0 + \frac{1}{u(t)}\int_{t_0}^{t} u(\tau)f(\tau)\,d\tau. \tag{2.3.4}$$

At this stage, you might think that this looks familiar... and it is! Basically, it is the same formula as (2.2.8); see Exercise 2.3.5.

Example 2.3.1. *Consider the differential equation*

$$\frac{dy}{dt} + 2y = 7.$$

Multiplying by u, we can rewrite this equation as

$$u\frac{dy}{dt} + y(-2u) = 7u. \tag{2.3.5}$$

We want u to satisfy

$$\frac{du}{dt} = -2u,$$

which is easy to arrange by choosing $u = e^{-2t}$. Thus (2.3.5) becomes

$$\frac{d}{dt}\left[e^{-2t}y\right] = 7e^{-2t}.$$

We can integrate this in order to obtain

$$e^{-2t}y(t) - y(0) = -\frac{7}{2}\left(e^{-2t} - 1\right).$$

Therefore solutions to the differential equation take the form

$$y(t) = e^{2t}y(0) - \frac{7}{2} + \frac{7}{2}e^{2t}.$$

Activity 2.3.1. *Use the method of integrating factors to find formulas for the following differential equations:*

1. $\frac{dy}{dt} + 2ty = t^2$
2. $\frac{dy}{dt} + y = \cos t$
3. $\frac{dy}{dt} + \cos t\, y = \cos t$
4. $\frac{dy}{dt} + y = t^2$

Activity 2.3.2. *Suppose a* $1/2$ *gallon bottle is being filled with water from a rusty pipe. Water coming out of the pipe at one gallon per minute. As the pipe gets flushed out, there is less rust coming out; we assume that the concentration of rust is* $0.01e^{-5t}$ *ounces per gallon. We furthermore assume that the bottle overflows so that it is always full of water and that there is perfect mixing. If there is* 0.05 *ounces of rust in the bottle, find a formula for how much rust is in the bottle as a function of time. At what time is there only* 0.0001 *ounces of rust?*

Exercise 2.3.1.

Find the general solution of the following equations:

1. $\frac{dy}{dt} = y + t^2$
2. $\frac{dy}{dt} = -2ty + e^{-t^2}$
3. $\frac{dy}{dt} = -\frac{y}{t} + \sin(t)$

Exercise 2.3.2. Solve the following initial value problems:

1. $\frac{dy}{dt} = y + \sin(t), \quad y(0) = 1$

2. $\frac{dy}{dt} = \frac{y}{t} + t, \quad y(1) = 0.$

Exercise 2.3.3. A 100-gallon mixing tank is full of pure water at time $t = 0$. Salty water of salt concentration 0.4 $^{\text{lb}}/_{\text{gal}}$ is being pumped into the tank at a decreasing rate of $e^{-0.05t}$ $^{\text{gal}}/_{\text{min}}$. The resulting salty water is also being drained from the tank so that its volume is kept constant at 100 gallons. Assuming the tank is always thoroughly mixed, find the amount of salt in the tank (in pounds) at time t. What will the concentration of salt roughly become in the long run?

Exercise 2.3.4. A 1 000-gallon tank is full of pure water. Salty water is being pumped into the tank at a decreasing rate of $\frac{1000}{10+t}$ gallons per hour; here the variable t denotes the number of hours since the beginning of the mixing process. The concentration of salt in the solution which is pumped into the tank is 0.01 pounds per gallon. The tank is constantly being mixed and drained so that the volume of the tank is maintained at 1 000 gallons. How much salt will there be in the tank in the long run?

Exercise 2.3.5. Let u be a solution to (2.3.3). Show that

$$P(t, s) = \frac{u(s)}{u(t)}$$

satisfies all the properties of the propagator function for r.