6.2 Constant coefficient homogeneous equations

Our plan for studying oscillator-type equations is the following. First, in this section, we study equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{6.2.1}$$

where a, b, c are constants. (We assume that $a \neq 0$ because otherwise we have a first-order equation.) Notice that the generic oscillator model (6.1.3) takes this form if the forcing is set to zero. In subsequent sections we introduce forcing terms on the right hand side.

Equations of the form (6.2.1) are called ***constant coefficient*** because the coefficients are constant, and are called ***homogeneous*** because the right side of the equation is zero.

One way to study the equation (6.2.1) is to express it as the first order system

$$\frac{dy}{dt} = v \qquad \frac{dv}{dt} = -\frac{c}{a}y - \frac{b}{a}v.$$

This can also be written in vector-matrix form as

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$-\lambda\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0,$$

which simplifies to

$$a\lambda^2 + b\lambda + c = 0. \tag{6.2.2}$$

Suppose now that λ is a solution to (6.2.2). The corresponding eigenvector must satisfy

$$\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} \bigstar \\ \clubsuit \end{pmatrix} = \lambda\begin{pmatrix} \bigstar \\ \clubsuit \end{pmatrix}.$$

In particular, we must have

$$\clubsuit = \lambda\bigstar$$

and therefore we may choose the eigenvector associated to λ to be

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

The associated eigensolution to the first order system is

$$e^{\lambda t}\begin{pmatrix}1\\ \lambda\end{pmatrix}.$$

From this we conclude the following: *If* λ *is a solution to* (6.2.2)*, then* $y(t) = e^{\lambda t}$ *is a solution to* (6.2.1).

In fact, we could have learned this a much easier way. If we plug the function $y(t) = e^{\lambda t}$ in to the equation (6.2.1), we obtain

$$e^{\lambda t}\left(a\lambda^2 + b\lambda + c\right) = 0. \tag{6.2.3}$$

Since $e^{\lambda t}$ is not the zero function, we again see that $e^{\lambda t}$ is a solution to (6.2.1) precisely when λ solves (6.2.2).

We now know how to construct "simple" solutions to (6.2.1). In order to use this to construct a general solution, we need to know that the superposition principle works for the equation (6.2.1). By direct computation, we can verify the following: *If* $y_1(t)$ *and* $y_2(t)$ *are solutions, then so is* $y(t) = \alpha y_1(t) + \beta y_2(t)$ *for any constants* α, β; see the exercises.

Since we now that the superposition principle works for the equation (6.2.1) we can now proceed as follows. If (6.2.2) has two real solutions λ_1 and λ_2, then we know that both

$$y_1(t) = e^{\lambda_1 t} \qquad \text{and} \qquad y_2(t) = e^{\lambda_2 t}$$

are solutions. Using the superposition principle, we find that the generic solution to (6.2.1) is

$$y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}.$$

If, however, the equation (6.2.2) has complex solutions $\lambda_\pm = a \pm bi$ then we can use the superposition principle to conclude that

$$y_1(t) = e^{at}\cos(bt) \qquad \text{and} \qquad y_2(t) = e^{at}\sin(bt)$$

are solutions. From this we deduce that a general solution to (6.2.1) is

$$y(t) = \alpha e^{at}\cos(bt) + \beta e^{at}\sin(bt);$$

see the exercises for details.

Example 6.2.1. *Let's find the general solution to the equation*

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

We see that $e^{\lambda t}$ *is a solution when*

$$0 = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3).$$

Thus both e^{-2t} *and* e^{-3t} *are solutions, and a general solution is*

$$y(t) = \alpha e^{-2t} + \beta e^{-3t}.$$

Notice that $y(t)$ *decays to zero without any oscillation as* $t \to \infty$.

While we can easily plot typical solutions $y(t)$ *on the* t–y *axis, we can also plot solutions in the* y–v *phase plane. Setting* $v = \frac{dy}{dt}$ *we have*

$$\begin{pmatrix} y \\ v \end{pmatrix} = \alpha e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

From this we see conclude that the phase portrait has a sink-type equilibrium at $(y, v) = (0, 0)$.

Example 6.2.2. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = 0.$$

We see that $e^{\lambda t}$ *is a solution when*

$$0 = \lambda^2 + \lambda + 6,$$

which occurs when

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{23}}{2} i.$$

Thus the general solution is

$$y(t) = \alpha e^{-t/2} \cos\left(\frac{\sqrt{23}}{2} t\right) + \beta e^{-t/2} \sin\left(\frac{\sqrt{23}}{2} t\right).$$

Notice that these solutions oscillate with exponentially decaying amplitude.

We now want to plot the solution in phase space. Since the eigenvalues λ *are complex with negative real part, the equilibrium at zero is a spiral sink. In order to determine the direction, we observe that* $v > 0$ *means that* y *is increasing. Thus the spiral is clockwise.*

If we wanted an exact formula for the trajectories in phase space, we can compute $v = \frac{dy}{dt}$, *from which we deduce that*

$$\begin{aligned}\begin{pmatrix} y \\ v \end{pmatrix} = \alpha &\left\{ e^{-t/2} \cos\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 0 \\ -\sqrt{23}/2 \end{pmatrix} \right\} \\ &+ \beta \left\{ e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 0 \\ \sqrt{23}/2 \end{pmatrix} \right\}.\end{aligned}$$

We conclude this section with some remarks about the superposition principle and the concept of linearity. Previously, we defined a first order system to be linear if it took the form

$$\frac{d}{dt}Y = F(Y)$$

for some linear function F. One consequence of this was the superposition principle: if $Y_1(t)$ and $Y_2(t)$ are solutions, then so is $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$ for any constants α, β.

We know that we can write (6.2.1) as a first order system. Thus one way to define what it means for a second order equation to be linear would be that the associated first order system is linear. This is a fine definition, but it is not the standard one. Rather, we make use of the superposition principle directly and say that (6.2.1) is defined to be ***linear*** because it has the property that if $y_1(t)$ and $y_2(t)$ are solutions, then so is $y(t) = \alpha y_1(t) + \beta y_2(t)$ for any constants α, β.

Exercise 6.2.1. Verify that the superposition principle works for the equation (6.2.1) as follows. Assume that $y_1(t)$ and $y_2(t)$ are solutions. Show by direct computation that this implies that $y(t) = \alpha y_1(t) + \beta y_2(t)$ is a solution for any constants α, β.

Exercise 6.2.2. Suppose $e^{\lambda t}$ is a solution to (6.2.1) with $\lambda = a + bi$. Show how to use this in order to conclude that

$$y_1(t) = e^{at}\cos(bt) \qquad \text{and} \qquad y_2(t) = e^{at}\sin(bt)$$

are solutions.

Exercise 6.2.3. Consider the second order differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

1. Find the general solution to this differential equation.

2. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Exercise 6.2.4. Show that the superposition principle also holds for ***non-constant coefficient homogeneous linear equations***, which take the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = 0$$

for functions a, b, c.

Exercise 6.2.5. Consider the differential equation

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0.$$

1. Find those values of α for which the function $y(t) = t^\alpha$ solves the differential equation.

2. Use the superposition principle from Exercise 6.2.4 to solve the IVP:

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0, \qquad y(1) = 2, \quad y'(1) = 4.$$

Exercise 6.2.6. Find the general solution of the following equations:

1. $\dfrac{d^2y}{dt^2} + \omega^2 y = 0;$

2. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0;$

3. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} = 0;$

4. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = 0;$

5. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0.$