3.4 Equilibrium points and linearization

Suppose we have a system of equations

$$\frac{dP}{dt} = f(P, R) \qquad \frac{dR}{dt} = g(P, R) \tag{3.4.1}$$

linearize:generi

and suppose that (P_*, R_*) is an equilibrium point of the system. Our goal is to find another system of equations, related to the system (3.4.1), such that

1. the new system closely approximates (3.4.1) when (P, R) is close to (P_*, R_*) and

2. the new system is simple enough that we can explicitly solve it.

If we can find such an approximating system, then we can use it to determine how solutions to (3.4.1) behave when they are close by to the equilibrium point. In particular, such an approximating system can be used to understand the stability of the equilibrium point.

The main tool we use is that of linear approximation. Recall that for a smooth function $f(x)$ we have

$$f(x) \approx f(x_*) + \frac{df}{dx}(x_*)\,(x - x_*) \qquad \text{when } x \approx x_*. \tag{3.4.2}$$

Taylor-1D

Example 3.4.1. *Suppose that $f(x) = \sqrt{x}$. If x is close to $x_* = 4$ then we have*

$$\sqrt{x} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}\,(x - 4) = 2 + \frac{1}{4}(x - 4).$$

In particular,

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2.025.$$

Compare this to the Sage output for $\sqrt{4.1}$, which is 2.02484567313166*. The approximation is pretty good!*

The linear approximation (3.4.2) is, of course, just the first-order Taylor approximation of the function f centered at $x = x_*$. For systems, we need the corresponding Taylor approximation for functions of two variables, which is

$$f(x, y) \approx f(x_*, y_*) + \frac{\partial f}{\partial x}(x_*, y_*)\,(x - x_*) + \frac{\partial f}{\partial y}(x_*, y_*)\,(y - y_*), \tag{3.4.3}$$

Taylor-2D

which holds when $(x, y) \approx (x_*, y_*)$.

We now apply (3.4.3) to the functions f and g appearing in (3.4.1). Notice that $f(P_*, R_*) = 0$ and $g(P_*, R_*) = 0$ because (P_*, R_*) is an equilibrium point. Thus we have

$$\begin{aligned} f(P,R) &\approx \frac{\partial f}{\partial P}(P_*, R_*)\,(P - P_*) + \frac{\partial f}{\partial R}(P_*, R_*)\,(R - R_*) \\ g(P,R) &\approx \frac{\partial g}{\partial P}(P_*, R_*)\,(P - P_*) + \frac{\partial g}{\partial R}(P_*, R_*)\,(R - R_*), \end{aligned}$$

when $(P, R) \approx (P_*, R_*)$. We also have

$$\frac{dP}{dt} = \frac{d}{dt}\,[P - P_*] \qquad \text{and} \qquad \frac{dR}{dt} = \frac{d}{dt}\,[R - R_*]\,,$$

simply because P_* and R_* are constants. Thus when (P, R) are close to the equilibrium point (P_*, R_*) we have

$$\begin{aligned} \frac{d}{dt}\,[P - P_*] &\approx \frac{\partial f}{\partial P}(P_*, R_*)\,(P - P_*) + \frac{\partial f}{\partial R}(P_*, R_*)\,(R - R_*) \\ \frac{d}{dt}\,[R - R_*] &\approx \frac{\partial g}{\partial P}(P_*, R_*)\,(P - P_*) + \frac{\partial g}{\partial R}(P_*, R_*)\,(R - R_*). \end{aligned}$$

We now make a change of variables, setting

$$v = P - P_* \qquad \text{and} \qquad w = R - R_*.$$

The functions v and w describe the displacement of (P, R) away from the equilibrium (P_*, R_*).

[picture needed]

With this change of variables, we obtain the ***linearization*** of (3.4.1) at (P_*, R_*):

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial f}{\partial P}(P_*, R_*)\,v + \frac{\partial f}{\partial R}(P_*, R_*)\,w \\ \frac{dw}{dt} &= \frac{\partial g}{\partial P}(P_*, R_*)\,v + \frac{\partial g}{\partial R}(P_*, R_*)\,w. \end{aligned} \qquad (3.4.4)$$

linearize:linearize

The system (3.4.4) is also called the "linearized equation" at (P_*, R_*).

Example 3.4.2. *Consider the predator-prey system*

$$\frac{dP}{dt} = P(1-P) - PR \qquad \frac{dR}{dt} = -R + 2PR.$$

We linearize this system about the equilibrium point $(P_*, R_*) = (.5, .5)$.

The system takes the form (3.4.1) *with*

$$f(P,R) = P(1-P) - PR = P - P^2 - PR,$$
$$g(P,R) = -R + 2PR.$$

We can easily compute

$$\frac{\partial f}{\partial P} = 1 - 2P - R \qquad \frac{\partial f}{\partial R} = -P$$
$$\frac{\partial f}{\partial P}(.5,.5) = -.5 \qquad \frac{\partial f}{\partial R}(.5,.5) = -.5$$

and

$$\frac{\partial g}{\partial P} = 2R \qquad \frac{\partial g}{\partial R} = -1 + 2P$$
$$\frac{\partial g}{\partial P}(.5,.5) = 1 \qquad \frac{\partial g}{\partial R}(.5,.5) = 0$$

Thus, setting $v = P - .5$ *and* $w = R - .5$, *we see that the linearized system is*

$$\frac{dv}{dt} = -.5v - .5w \qquad \frac{dw}{dt} = v$$

`FindThemAll`

Exercise 3.4.1. Find all equilibrium solutions of the following system of differential equations:

$$\frac{dx}{dt} = 2 + 14x + 12y, \qquad \frac{dy}{dt} = -9 + 12x + 21y.$$

Then linearize the system about each equilibrium.

`EqSol-Last`

Exercise 3.4.2. Find all equilibrium solutions of the predator-prey model:

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy, \qquad \frac{dy}{dt} = 5y\left(1 - \frac{y}{5}\right) - 4xy.$$

Then linearize the system about each equilibrium.