2.4 Slope fields

The previous three sections show that it is possible to construct formulas for solutions to differential equations of the form

$$\frac{dy}{dt} = h(y)g(t) \quad \text{and} \quad \frac{dy}{dt} = r(t)y + f(t).$$

In general, however, we do not expect to be able to obtain formulas for solutions to differential equations. Nevertheless, it is possible to deduce many features of solutions to differential equations without being able to find expressions for those solutions.

In fact, we have already seen an example of this. Consider a differential equation of the form

$$\frac{dy}{dt} = f(t, y),$$

where the function f is globally smooth. Suppose that y_* is an equilibrium solution. If $y(t)$ is some other solution such that $y(0) > y_*$, then we know that $y(t) > y_*$ so long as the solution $y(t)$ exists. Graphically, we interpret this as stating that in the t–y plane, the graph of the solution $y(t)$ cannot cross the line $y = y_*$. That is, equilibrium solutions provide barriers that other solutions cannot cross.

Here we recall Example 1.5.1

Example 2.4.1. *The differential equation*

$$\frac{dy}{dt} = y(1 - y) \tag{2.4.1}$$

has equilibrium solutions $y_* = 0$ *and* $y_* = 1$*. Thus for any solution* $y(t)$ *we know that*

- *if* $y(0) < 0$*; then* $y(t) < 0$ *so long as the solution exists;*
- *if* $0 < y(0) < 1$*, then* $0 < y(t) < 1$ *and, since the solution cannot blow up,* $y(t)$ *exists for all* t*;*
- *if* $y(0) > 1$*, then* $y(t) > 1$ *so long as the solution exists.*

We can actually learn more about solutions to differential equations by studying equilibrium solutions. First, however, we introduce a graphical tool for understanding solutions to differential equations.

Consider, for a moment, the differential equation in Example 2.4.1. If at some time t we happen to know the value of a solution $y(t)$, then the differential equation tells us the derivative of the solution at that time. For example, if we have a solution $y(t)$ such that $y(1) = 0.3$, then the differential equation tells us that

$$y'(1) = y(1)\,(1 - y(1)) = 0.3\,(1 - 0.3) = 0.21.$$

In fact, we know from the Fundamental Theorem of ODEs that there does exist a solution $y(t)$ such that $y(1) = 0.3$; from the differential equation we learn the what the slope of that solution is.

Consider now some other solution $y(t)$, say with $y(1) = 0.1$. The differential equation tells us that for this solution we have $y'(1) = 0.1(1-0.1) = .09$. We can continue on this way, considering a variety of solutions, and end up with the following table:

t	$y(t)$	$y'(t)$
1	0	0
1	0.1	0.09
1	0.2	0.16
1	0.3	0.21
1	0.4	0.24
1	0.5	0.25
etc.		

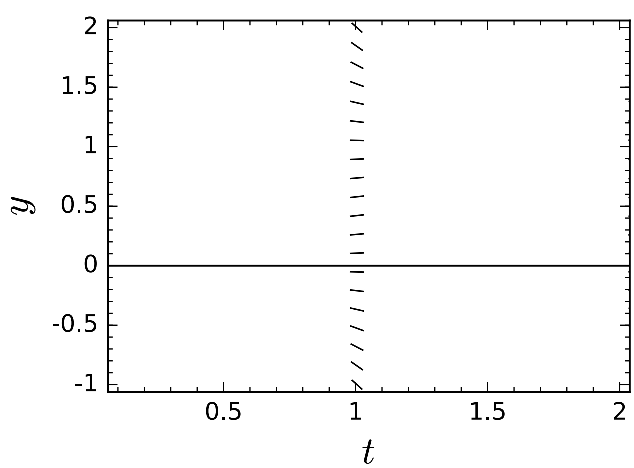

Figure 2.4.1: Plot of the slopes of solutions to (2.4.1) at time $t = 1$.

What do all these numbers tell us? One way to understand them is to interpret them in terms of the graphs of the solutions. The derivative tells us the slope of the graph in the t–y plane, so this list of numbers tells us the slopes of various solutions at time $t = 1$ and with various values of y. If we plot these little slopes, we end up with a picture as in Figure 2.4.1.

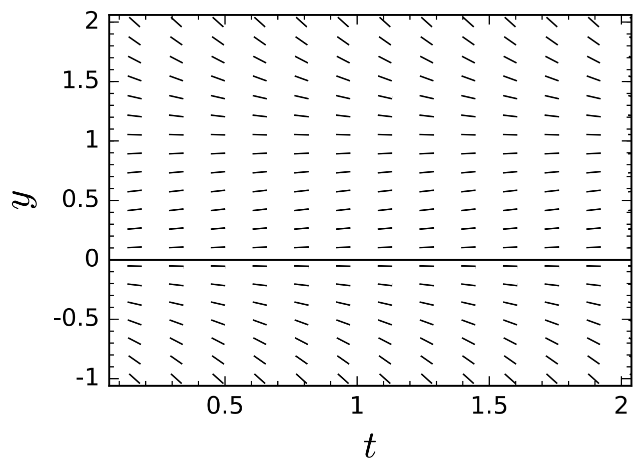

Figure 2.4.2: Plot of the slopes of solutions to (2.4.1)

Of course, there is no reason to focus only on $t = 1$. For any value of t and y we can use the differential equation to compute the slope of the solution passing through that point. Plotting these slopes at each point for the differential equation (2.4.1) yields the picture in Figure 2.4.2.

The plot in Figure 2.4.2 called the ***slope field plot*** for the differential equation (2.4.1). Since the slope field plot shows the slopes of all solutions, we can use the plot to easily imagine how solutions to the differential equation behave – they "follow" the slope field.

Example 2.4.2. *Consider the differential equation*

$$\frac{dy}{dt} = -2y. \tag{2.4.2}$$

We know from previous work that solutions take the form $y(t) = Ce^{-2t}$.

Figure 2.4.3 shows a plot of the slope field for (2.4.2) *together with the plots of two different solutions.*

Let's now return to the equation (2.4.1). We know that there are two equilibrium solutions $y_* = 0$ and $y_* = 1$, which other solutions cannot cross.

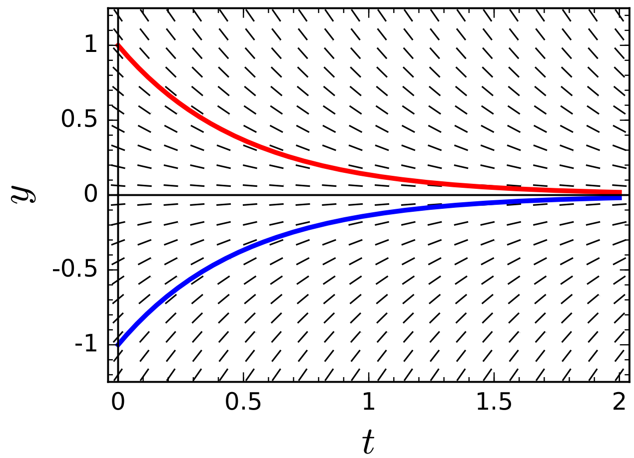

Figure 2.4.3: The slope field for (2.4.2) with the plots of two solutions superimposed on it. Notice that the solution $y(t) = e^{2t}$ (in red) and the solution $y(t) = -e^{-2t}$ (in blue) both follow the slope field lines.

Suppose we have a solution y that, at some time t has $y(t)$ between 0 and 1. Then the equation tells us that the slope of the graph at that point is

$$\frac{dy}{dt} = \underbrace{y(t)}_{>0}\underbrace{(1 - y(t))}_{>0} > 0.$$

Thus whenever the value of a solution y is between 0 and 1, the slope of the graph of solutions is positive. Furthermore, whenever the value of y is greater than 1 we have

$$\frac{dy}{dt} = \underbrace{y}_{>0}\underbrace{(1 - y)}_{<0} < 0$$

and the solution is decreasing. Thus we can systematically deduce the following:

- Solutions $y(t)$ satisfying $y(t) < 0$ are always decreasing, and thus moving away from the equilibrium $y_* = 0$.
- Solutions $y(t)$ satisfying $0 < y(t) < 1$ are always increasing, and thus moving away from the equilibrium $y_* = 0$ and towards the equilibrium $y_* = 1$.

- Solutions $y(t)$ satisfying $y(t) > 1$ are always decreasing, and thus moving towards the equilibrium $y_* = 1$.

In particular, we see that if a solution $y(t)$ satisfies $y(0) > 0$, then $y(t) \to 1$ as $t \to \infty$.

(Technical point: We know that if $y(0) > 1$, then the solution $y(t)$ is always decreasing. How do we know that the limit is actually $y(t) \to 1$? What prevents the solution from having an asymptote at some value other than 1?)

If we have an equilibrium point such that *all* nearby solutions move towards the equilibrium value, then we call that equilibrium point a ***stable*** equilibrium point. An equilibrium point where at least one nearby solution moves away from the equilibrium value is called an ***unstable*** equilibrium point. Thus for the equation (2.4.1), the equilibrium point $y_* = 0$ is unstable, while $y_* = 1$ is stable.

The stability of equilibrium points is important in applications: In "the real world" we do not expect equilibrium situations to be perfectly achieved. Thus if a system is to remain "in equilibrium" it must be the case that the equilibrium is stable. In other words, we only expect to observe systems in stable equilibrium configurations, not unstable ones.

The process of identifying the equilibrium points of an equation and determining their stability is a key part of the *qualitative analysis* of a differential equation.

Example 2.4.3. *Perform a qualitative analysis of the differential equation*

$$\frac{dy}{dt} = y^2(1 - y^2). \tag{2.4.3}$$

The equilibrium solutions are $y_* = 0$, $y_* = 1$, and $y_* = -1$. The slope field plot appears in Figure 2.4.4

From this we deduce that $y_* = 0$ and $y_* = -1$ are unstable, while $y_* = 1$ is stable.

We can easily use computers to generate slope field plots. For instance, the Sage code used to generate the plot in Figure 2.4.2 is

```
var('t,y')
slopePlot = plot_slope_field(y*(1-y), (t,0,2), (y,-1,2),
    axes_labels=['$t$','$y$'])
slopePlot.show()
```

The code used to generate the plot in Figure 2.4.3 is

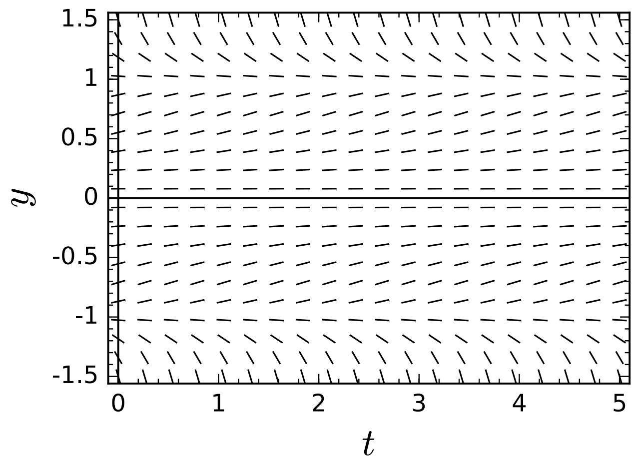

Figure 2.4.4: The slopefield for the differential equation (2.4.3).

```
var('t,y')
slopePlot = plot_slope_field(-2*y, (t,0,2), (y,-1.2,1.2),
    axes_labels=['$t$','$y$'])
solnPlot1 = plot(exp(-2*t),(t,0,2),color='red',thickness=2)
solnPlot2 = plot(-exp(-2*t),(t,0,2),color='blue',thickness
    =2)
mainPlot = slopePlot + solnPlot1 + solnPlot2
mainPlot.show()
```

In order to have Sage generate a `pdf` file containing the graphic, simply replace `mainPlot.show()` by the command

```
mainPlot.save(filename="basic-model-slope-plot.pdf",figsize
    =[4,3])
```

Activity 2.4.1. *Consider the differential equation*

$$\frac{dy}{dt} = cos(y).$$

Find the equilibrium solutions and determine their stability.

Exercise 2.4.1.

For each differential equation below you need to

- Find a formula for several "typical" solutions,

- Make plots (using Sage) of these solutions,
- Generate a slope field plot (using Sage), and
- Superimpose your solution plots on top of the slope field plot.

Turn in printouts of the final, combined plot. You should also write a sentence or two describing the long-term behavior of solutions.

1. $\frac{dy}{dt} = -y + t + 1;$
2. $\frac{dy}{dt} = y(9 - y^2);$
3. $\frac{dy}{dt} = y + 2t.$

Exercise 2.4.2. Without using 'technology', match the plots of solutions in Figure 2.4.5 to the corresponding differential equation:

1. $\frac{dy}{dt} = (y - 1)(y - 2e^t)$
2. $\frac{dy}{dt} = (y + 1)(y + e^t)$
3. $\frac{dy}{dt} = y\left(1 - \frac{y}{2-\sin(t)}\right)$
4. $\frac{dy}{dt} = y + \sin(t).$

Exercise 2.4.3. The size of the fish population of a lake is modeled by a logistic equation. It is known that under **ideal** conditions the per capita growth rate of fish is 80%. It is also known that the lake has the carrying capacity of 20 000 fish. It is estimated that currently this lake has 15 000 fish. The owners of the lake are considering harvesting.

1. Write down the logistic equation which models the size of the fish population. What is the current per capita growth rate of the population?
2. Modify the logistic equation to accommodate for (gradual) annual harvesting at the relative rate of 60%. Perform the qualitative analysis of your modified equation without any use of "technology". Then check your predictions with Sage (and show the print-out).

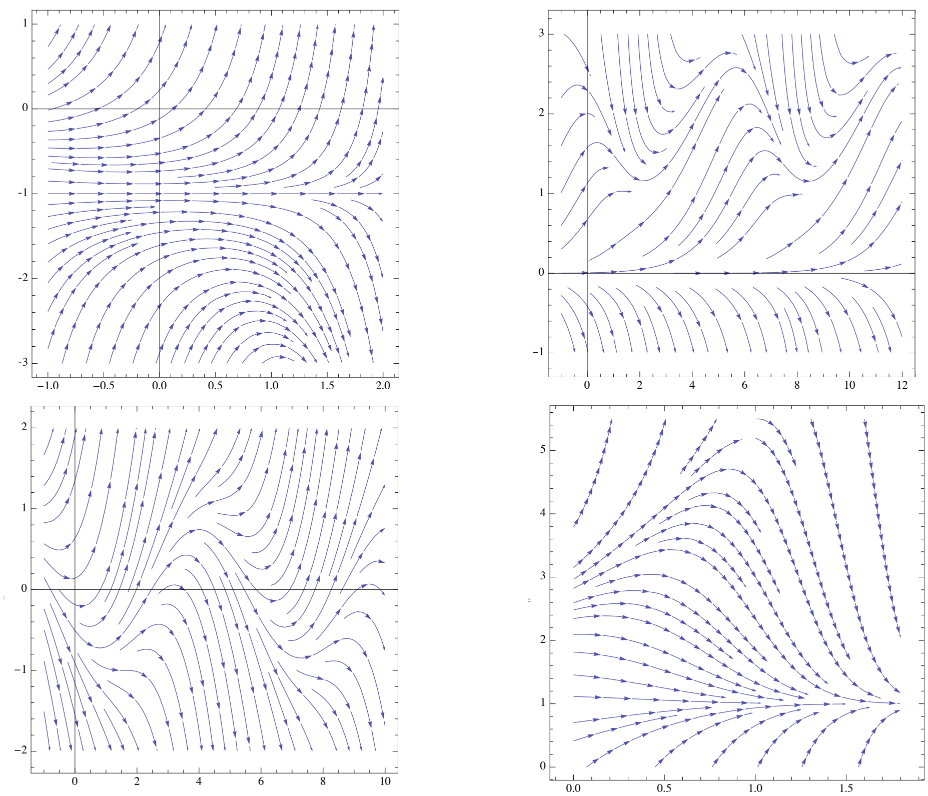

Figure 2.4.5: Plots showing various solutions to differential equations.

Exercise 2.4.4. Consider the IVP

$$\frac{dy}{dt} = -y^3 + y^2 + 12y, \qquad y(0) = 1.$$

Perform the qualitative analysis of the differential equation, and based on your conclusions make an educated guess about the long-term fate of the solution of our IVP. You should check your answer with Sage (and turn in the print out of the slope field..

Exercise 2.4.5. Consider the differential equation

$$\frac{dy}{dt} = y\left(\frac{y}{2e^t} - 1\right)$$

One can easily see that $y_1(t) = 0$ and $y_2(t) = 4e^t$ are both solutions of this differential equation.

1. Produce a rough sketch of the slope field corresponding to this equation **without** using "technology". Pay particular attention to the slope field along the curve $y = 2e^t$, between the curves $y = 2e^t$ and $y = 4e^t$, between $y = 2e^t$ and $y = 0$, above the curve $y = 4e^t$ and below the line $y = 0$. Check your answer on Mathematica.

2. A certain variable quantity $y(t)$ satisfies the differential equation in this problem. It is known that $y(0) \approx 4$, but the exact value of $y(0)$ is not known. What can you say about the long term behavior of this quantity i.e. about $\lim_{t \to +\infty} y(t)$?