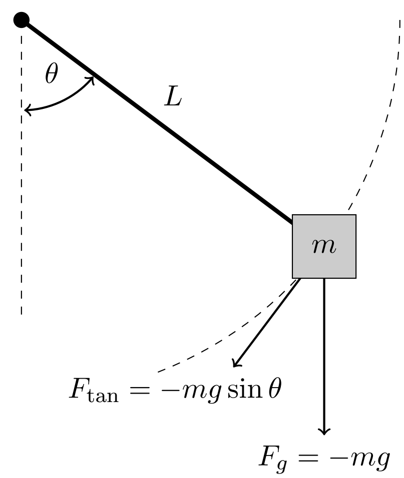
7.4 The nonlinear pendulum

In this section we consider a classic example: the nonlinear pendulum. We suppose that a weight of mass m is affixed to the end of a rigid and lightweight rod having length L. The other end of the rod is fixed in such a way that the rod can rotate freely in a vertical plane. We describe the motion of the mass in terms of the angle that the rod makes with the downward direction.

We assume that the only force acting on the mass is due to gravitation; near the surface of the earth we use the formula that this force has magnitude $F_g = -mg$ in the vertical direction. Here $g \approx 10$ m/s is Newton's "little" gravitational constant. Since the rod is rigid, the only motion of the mass permitted is tangental to the circle of radius L centered at the fixed endpoint of the rod. The tangential component of the gravitational force, which we denote $F_{\tan}$, is given by $F_{\tan} = -mg\sin\theta$. This situation is described by the following figure:



We now derive a differential equation for $\theta(t)$ using Newton's second law. The tangential acceleration of the mass is given by

$$\frac{d^2}{dt^2}\left[L\theta\right].$$

Thus the tangential component of the relation $ma = F$ becomes

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta,$$

which we rewrite as

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta. \tag{7.4.1}$$

We can write (7.4.1) as a Hamiltonian system, by letting $\omega = \frac{d\theta}{dt}$. The resulting system is

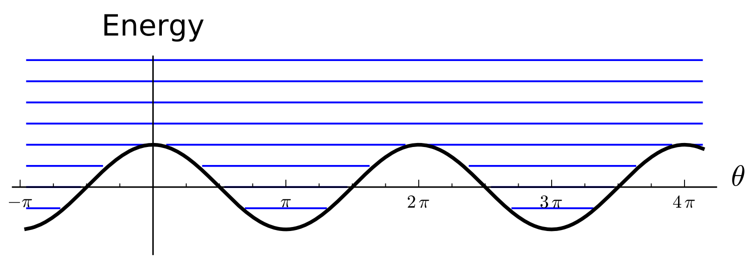
$$\frac{d\theta}{dt} = \omega \qquad \frac{d\omega}{dt} = -\frac{g}{L}\sin\theta. \tag{7.4.2}$$

A potential function for (7.4.2) is $V(\theta) = \frac{g}{L}\cos\theta$ and therefore the quanitity

$$H = \frac{1}{2}\omega^2 + \frac{g}{L}\cos\theta$$

is conserved for solutions to (7.4.2).

Using this conserved quantity, we find that the energy plot for (7.4.2) is



Activity 7.4.1. *In this energy diagram, the unknown θ can have any possible real value. How should we interpret θ larger than 2π?*

Using the energy diagram above, we see that there are four types of solutions to (7.4.2):

- Solutions with total energy $H = -g/L$ are equilibrium solutions corresponding to θ being a multiple of 2π, which means that the mass is simply "hanging" straight down. This is a center-type equilibrium.

- Solutions with energy $-g/L < H < g/L$ are trapped/bound solutions that oscillate about the center-type equilibrium.

- Solutions with $H = g/L$ are either at the saddle-type equilibrium points where θ is an odd multiple of π, or are headed towards such an equilibrium. Physically, these equilibria correspond to the mass "perfectly balanced" in a position directly above the spot where the rod is fixed. It is not surprising to us to see that this is an unstable (saddle-type) equilibrium situation.

- High energy solutions with $H > g/L$ have $\theta \to \pm\infty$ as t grows. These correspond to the pendulum "spinning" about the spot where the rod is fixed.

Activity 7.4.2. *Suppose we start out with $\theta(0) = 0$. What does the initial angular velocity $\omega(0)$ need to be in order for the solution to be a "high-energy" solution?*

Activity 7.4.3. *Use the energy plot above to draw the phase portrait for* (7.4.2)*.*

Exercise 7.4.1. Find an equilibrium point of the system (7.4.2) which, based on the energy diagram, is a center.

- Linearize the system about this equilibrium and show that indeed it is a center by finding the eigenvalues.

- Compare the linearized system to that of the simple harmonic oscillator.

- Solve the simple harmonic oscillator equation in order to find a relationship between L and the frequency of the oscillations.

- Suppose you want to make a pendulum clock, where the pendulum oscillates with a period of one second. How long should L be?

Exercise 7.4.2. Find an equilibrium point of the system (7.4.2) which, based on the phase portrait, seems to be unstable. Linearize the system about this equilibrium and show that indeed the equilibrium point is unstable.

Exercise 7.4.3. Let's take a look at the units in this problem. The angular variable θ has no units, while we measure length in meters, time in seconds, and the mass m in kilograms. The gravitational constant $g \approx 10\ \mathrm{m/s^2}$.

1. What units does ω have?

2. Explain why $\frac{1}{2}mL^2\omega^2$ has the "usual" units of kinetic energy.

3. Explain why the "correct" potential energy V should be $V(\theta) = mgL\cos\theta$, and why this has the appropriate units.

7.5 Hamiltonian systems with damping

In this section we consider the effects of adding a linear damping term to a Hamiltonian system. Suppose we have a Hamiltonian system

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -V'(x).$$

We can modify the system to include a linear damping term, obtaining

$$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = -V'(x) - bv, \tag{7.5.1}$$

where $b > 0$ is a constant.

Solution to (7.5.1) do not have a conserved quantity. However, we can still make use of the energy

$$H = \frac{1}{2}mv^2 + V(x)$$

in order to study solution to (7.5.1). To do this, we suppose that x, v is a solution to (7.5.1) and compute

$$\frac{d}{dt}H = mv\frac{dv}{dt} + V'(x)\frac{dx}{dt} = -bv^2.$$

Thus we see that

$$\frac{d}{dt}H \leq 0,$$

with equality precisely when $v = 0$. In particular, H is non-increasing and is in fact decreasing except at moments when $v = 0$.

Quantities that are either non-increasing or non-decreasing are called ***monotone*** quantities. Thus we see that the energy H is a monotone quantity for (7.5.1).

The usefulness of monotone quantities is similar to the utility of conserved quantities. Because H is monotone decreasing, we know that solution to (7.5.1) never move upward in an energy diagram, and in fact move downward except at moments when $v = 0$.

In order to see how this works in practice, let's consider an example for which we can obtain an exact formula for the solution.

Example 7.5.1. *Consider the damped oscillator equation*

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0.$$

We can write this equation as the first order system

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -x - bv. \tag{7.5.2}$$

We know that the phase portrait for this system has a spiral-sink type equilibrium at $(x, v) = (0, 0)$*. We now explore what a spiral sink looks like in an energy diagram.*

The energy for (7.5.2) *is*

$$H = \frac{1}{2}v^2 + \frac{1}{2}x^2.$$

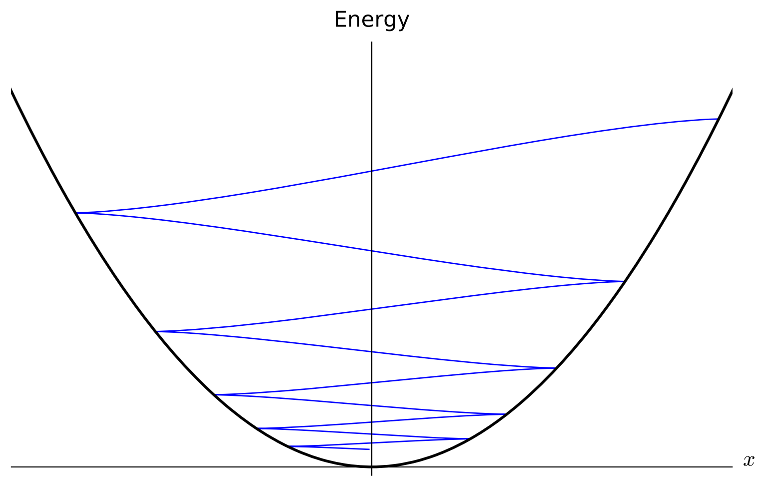
Through direct computation, we find that for solutions to (7.5.2) *we have*

$$\frac{d}{dt}H = -bv^2.$$

Thus the energy for the solution is decreasing, except when $v = 0$*.*

Consider a solution to (7.5.2) *for which* $v(0) = 0$ *and* $x(0) > 0$*, as depicted in the energy diagram. Because* $v(0) = 0$*, the trajectory of the solution starts out on the potential curve. At that instant* H *is not decreasing and the solution begins moving to the left. Immediately, the velocity becomes nonzero and the energy of the solution begins to drop. As the* x *approaches zero, the solution approaches its largest value of* v*, which now corresponds to the largest rate of decrease in energy. After the trajectory passes through* $x = 0$ *the rate at which the energy is dropping decreases until the solution eventually meets the graph of the potential function. At the point where the solution trajectory meets the potential curve, the velocity is zero and the rate of energy loss is instantaneously zero. The solution then reflects off the potential curve and* x *begins to increase. The whole scenario described above repeats itself, with the result being that the solution trajectory "bounced" down the "potential well" towards the equilibrium at the bottom.*

The following picture, for which $b = 0.1$*,* $x(0) = 1$*, and* $v(0) = 0$*, illustrates this behavior:*



In general, whenever we have a potential function $V(x)$ with a local minimum, we expect solutions to (7.5.1) that are close to the local minimum to exhibit behavior similar to what we saw in Example 7.5.1. In order to build intuition for how solutions behave in the vicinity of a local maximum, we consider the following.

Activity 7.5.1. *Consider the potential function $V(x) = -\frac{1}{2}x^2$ for a particle with mass $m = 4$. The corresponding damped Hamiltonian system, with damping constant $b = 3$, is*

$$\frac{dx}{dt} = v \qquad 4\frac{dv}{dt} = x - 3v.$$

Analyze this system via the following.

1. *Find the general solution and sketch the phase portrait.*
2. *Consider the solution with $x(0) = -1$ and $v(0) = 1$. Draw the trajectory of this solution in the phase portrait. Then draw the trajectory of this solution in the energy diagram.*
3. *Consider a solution with $x(0) = -1$ and $v(0) > 1$. Draw the trajectory in both the phase portrait and energy diagram.*
4. *Consider a solution with $x(0) = -1$ and $v(0) < 1$. Draw the trajectory in both the phase portrait and energy diagram.*

What can you conclude generally about solutions to this system?

From the previous activity, we see that if a potential function has a local maximum, then that maximum acts as a sort of barrier. Some solutions will have enough energy to pass over the barrier, while other solutions will not have enough energy to pass over. Typically, those solutions with insufficient energy to pass will reflect off the potential. However, there will be one trajectory corresponding to solutions that precisely "land" on top of the local maximum. In the phase portrait, these are the "lucky" solutions that approach the saddle point equilibrium.

Activity 7.5.2. *Consider the potential $V(x) = x(1 - x^2)$. Describe the behavior of solutions to the corresponding damped Hamiltonian system. Illustrate using an energy diagram.*

Activity 7.5.3. *Consider the case of the nonlinear pendulum from the previous section, but now with the addition of damping. Describe the behavior of solutions. Illustrate using an energy diagram.*

Exercise 7.5.1. The equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 3x^2 = 3$$

can be perceived as a model for some non-linear oscillation with friction (damping).

1. Write down the corresponding frictionless (undamped) system. What is the conserved energy for that system?

2. Show that the energy for the damped system is monotone decreasing.

3. Draw the energy diagram for the damped system.

4. Then discuss the possibilities for the long-term fate of the solutions.

Exercise 7.5.2. Repeat the previous problem for the equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 3x^2 = -12$$

Writing assignment 5: Gravitation.

In this report you explore some simple equations describing the motion of a small planet of mass m as it moves relative to a large star of mass M. We assume the planet moves in the equatorial plane of the planet, so that we can describe the motion of the planet by the functions

$r(t)$, describing the distance from the star to the planet, and

$\theta(t)$, describing the angular motion of the planet.

According to our friends in the Physics Department, the relevant equations are:

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 - \frac{GMm}{r^2} \tag{7.5.3}$$

$$\frac{d}{dt}\left[mr^2\frac{d\theta}{dt}\right] = 0 \tag{7.5.4}$$

Here G is Newton's gravitational constant: $G \approx 6.673 \times 10^{-11}\ ^{\text{N}\cdot\text{m}^2}/_{\text{kg}^2}$ You should take these equations to be physics-given 'facts.'

Your task in this report is to investigate the behavior of the planet predicted by these equations. Before you begin, let me make some remarks:

1. The equation (7.5.4) implies that the quantity $L = mr^2\frac{d\theta}{dt}$ is conserved; L is called the angular momentum. The fact that it is conserved, means that we can treat L as a parameter (just as we treat the energy as a parameter when studying SHO).

2. Knowing that $L = mr^2\frac{d\theta}{dt}$ is conserved means that we can replace $\frac{d\theta}{dt}$ by $\frac{L}{mr^2}$ in (7.5.3). The result is an equation which only involves r, though it does have four constant parameters: G, M, m, L. Your plan should be to focus on this "reduced" equation for r because once r is determined, θ can be determined by integration.

3. Although there are four parameters (G, M, m, L), in the reduced equation they only come in two combinations: $a = GM$ and $b = L^2/m^2$. (Note that both a and b are non-negative.) Thus when you are exploring this system, you really only need to think about the two parameters a and b.

4. Finally, since r represents the distance from the star to the planet, we are only interested in the case that $r \geq 0$. (What does it mean for $r \to 0$? What about $r \to \infty$?)

Your report is supposed to be an all-inclusive analysis of the behavior of solutions to the system, based on analyzing the reduced equation for r. As part of your report you should:

1. Explain how to start with the system (7.5.3)-(7.5.4) and obtain a reduced equation for r that has two parameters, a and b. Explain how to (physically) interpret the limits $a \to 0$ and $b \to 0$.
2. Show that the reduced equation is actually a Hamiltonian system for some potential function V.
3. Use the potential function to analyze completely the reduced equation, describing possible scenarios, etc.
4. What happens to the potential function in the limit as $a \to 0$ or $b \to 0$? What is the resulting behavior of r?