

## Chapter 7

# Conservations laws and Hamiltonian systems

In this chapter we study differential equations using the concept of “conservation of energy.” Before we begin, it is worth taking a moment to consider what we mean by this concept. We quote Richard Feynman from his *Lectures on Physics*:

*[Conservation of energy] states that there is a certain quantity, which we call energy, that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.*

–<http://www.feynmanlectures.caltech.edu/I.04.html>

You are encouraged to follow the link above and read the subsequent analogy.

### 7.1 Conservation of energy

Suppose we have a particle moving along the  $x$  axis. We describe the motion of the particle with a function  $x(t)$  that tells us the location at time  $t$ . Physicists have devised two quantities that describe the energy of such a particle:

- The **kinetic energy**  $K$  is the energy associated to the motion of the particle. The kinetic energy is given by the formula

$$K = \frac{1}{2}m \left( \frac{dx}{dt} \right)^2.$$

If we set  $v = \frac{dx}{dt}$  and  $p = mv$ , then the formula for the kinetic energy can be written as either

$$K = \frac{1}{2}mv^2 \quad \text{or} \quad K = \frac{p^2}{2m}.$$

- The **potential energy**  $V$  is the energy associated to the particle being at a particular location. Consequently, we typically write the potential energy as function  $V(x)$ , where  $x$  is the location of the particle.

The **total energy**  $H$  is given by

$$H = K + V = \frac{1}{2}mv^2 + V(x). \quad (7.1.1)$$

Note that the quantity  $H$  is really a function of depends on both  $x(t)$ , as well its first derivative  $x'(t)$ . If we want to explicitly indicate this, we can write

$$H[x(t)] = \frac{1}{2}m (x'(t))^2 + V(x(t)),$$

or simply

$$H[x] = \frac{1}{2}m \left( \frac{dx}{dt} \right)^2 + V(x).$$

While the notation  $H[x]$  is useful because it reminds us that for each function  $x$  we obtain a different quantity  $H[x]$ , we will typically suppress the dependence  $x$  and simply write ‘ $H$ ’.

Since  $H$  depends on  $x$ , and  $x$  depends on  $t$ , the energy  $H$  can also be viewed as a function of  $t$ . We say that the energy  $H$  is **conserved for function  $x$**  if  $H$  is a constant function in time. We can mathematically express this **conservation of energy** by

$$\frac{d}{dt}H = 0.$$

In classical Newtonian physics, the trajectory of a particle is determined by a differential equation arising from Newton’s Second Law,

$$m \frac{d^2x}{dt^2} = F, \quad (7.1.2)$$

where  $F$  is an expression describing the various “forces” acting on the particle. While Newton’s Second Law is expressed as a second-order equation, it will be more convenient for us to consider its first-order formulation

$$\frac{dx}{dt} = v \quad m \frac{dv}{dt} = F. \quad (7.1.3)$$

**Example 7.1.1.** *It is useful to keep in mind the example, introduced in §6.1, of the simple harmonic oscillator equation*

$$m \frac{d^2x}{dt^2} = -kx, \quad (7.1.4)$$

which can be written in first-order form as

$$\frac{dx}{dt} = v \quad m \frac{dv}{dt} = -kx; \quad (7.1.5)$$

here  $k$  is some positive constant describing the “strength” of the oscillator.

Given a differential equation of the form (7.1.3), it is natural to ask whether energy is conserved by solutions to that equation. Since the expression for the energy  $H$  depends on the potential function  $V(x)$ , this question can be reformulated as follows:

*Suppose we have a differential equation of the form (7.1.3). Does there exist a potential function  $V(x)$  such that the corresponding energy  $H = \frac{1}{2}mv^2 + V(x)$  is conserved by solutions to the differential equation?*

In order to address this question, we apply  $\frac{d}{dt}$  to  $H$ , computing (using the chain rule)

$$\frac{d}{dt}H = mv \frac{dv}{dt} + V'(x) \frac{dx}{dt}.$$

Using that  $v = \frac{dx}{dt}$  we see that  $\frac{d}{dt}H = 0$  precisely when

$$0 = v \left( m \frac{dv}{dt} + V'(x) \right).$$

Thus one way to ensure that the energy (7.1.1) is conserved is to require that

$$m \frac{dv}{dt} + V'(x) = 0.$$

In other words, if  $x$  and  $v$  satisfy the differential equation

$$\frac{dx}{dt} = v \quad m \frac{dv}{dt} = -V'(x) \quad (7.1.6)$$

then the energy (7.1.1) is conserved.

The system (7.1.6) is called the **Hamiltonian system** for potential function  $V(x)$ . Hamiltonian systems are precisely those systems for which solutions have a conserved energy. Thus if solutions to (7.1.3) then we see that we must have

$$F = -V'(x)$$

for some potential function  $V(x)$ .

**Example 7.1.2.** Consider the simple harmonic oscillator equations (7.1.5). This system takes the form (7.1.6) with potential function

$$V(x) = \frac{1}{2}kx^2.$$

Thus if  $x, v$  are a solution to (7.1.5), then the energy

$$H = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

is constant.

**Activity 7.1.1.** Consider the potential function  $V(x) = -(x - 2)^2(x - 5)$ . Find the Hamiltonian system for which the energy

$$H = \frac{1}{2}v^2 - (x - 2)^2(x - 5)$$

is conserved.

**Activity 7.1.2.** Consider the second order equation

$$\frac{d^2x}{dt^2} = -x^2.$$

Show that the corresponding first-order system is a Hamiltonian system. What is a conserved energy for this system?

Notice that when completing Activity 7.1.2, you actually had some freedom when constructing the potential function. If we are given a differential equation that can be written in the form of a Hamiltonian system, then the potential function is not uniquely determined. We can always add a constant to the potential function and obtain a conserved energy. Mathematically, this is simply a consequence of the fact that the derivative of a constant is zero, which implies that if any particular energy is conserved, then so is that energy plus a constant. In physics, the fact that one can always add a

constant to the potential function is in indication that it is the “potential difference” or the relative (rather than absolute) value of the potential that is “physically meaningful.”

Thus far we have established that solutions to Hamiltonian systems of the form (7.1.6) obey conservation energy. We now show how to use this conservation of energy in order to plot the trajectory of solutions in phase space.

First, we note that because the energy  $H$  depends only on  $v$  and  $x$ , we can determine the energy of a solution from the initial conditions

$$x(0) = x_0 \quad v(0) = v_0.$$

In particular, the energy of a solution with these initial condition is

$$H = \frac{1}{2}mv_0^2 + V(x_0).$$

Once we know the energy  $H$  of a given solution, then the conservation of energy implies that at all times we have

$$H = \frac{1}{2}mv^2 + V(x).$$

This implicitly tells us the trajectory that the solution takes through phase space.

**Example 7.1.3.** *Suppose  $x, v$  is a solution to the simple harmonic oscillator system (7.1.5) with initial conditions*

$$x(0) = x_0, \quad v(0) = v_0.$$

*The energy of this solution is*

$$H = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2. \quad (7.1.7)$$

*At all future times, the solution satisfies*

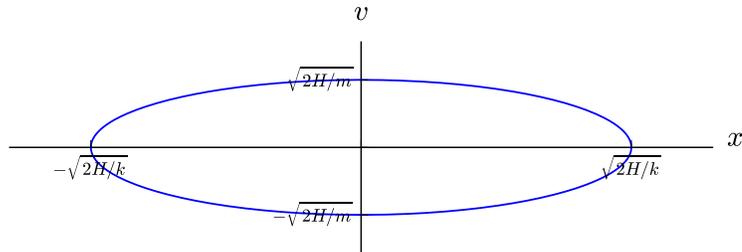
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2,$$

*which we rewrite as*

$$v^2 + \frac{k}{m}x^2 = v_0^2 + \frac{k}{m}x_0^2.$$

*Thus the solution traverses an ellipse in the phase plane.*

*With  $H$  as in (7.1.7), the trajectory is as shown in the following diagram:*



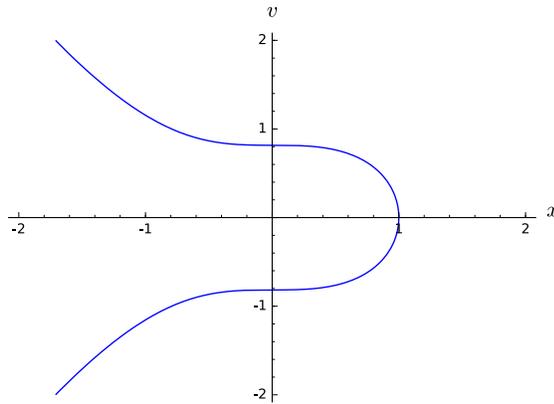
The spatial maximal extent of the solution is  $x = \pm\sqrt{2H/k}$ , while the maximal velocities achieved by the solution are  $v = \pm\sqrt{2H/m}$ .

**Activity 7.1.3.** Suppose  $x$  is the solution to the equation in Activity 7.1.2 having initial conditions

$$x(0) = 1, \quad x'(0) = 0.$$

What is the trajectory that the solution will follow in phase space?

You should find that the trajectory looks like the following:



We conclude this section with examples of systems that are *not* Hamiltonian systems.

**Example 7.1.4.**

1. Oscillator equations with a non-zero damping term are not Hamiltonian systems. To see this, recall that the first-order system corresponding to the generic oscillator equation is

$$\frac{dx}{dt} = v \quad m \frac{dv}{dt} = -bv - kx.$$

Since the right side of the second equation is not a function of  $x$  alone it does not take the form (??).

2. Oscillator equations with time-dependent forcing are not Hamiltonian systems. To see this, we note that the first-order system for the forced simple harmonic oscillator is

$$\frac{dx}{dt} = v \quad m \frac{dv}{dt} = -kx + f.$$

If  $f$  is a function of  $t$ , then the right side of the second equation is not a function of  $x$  alone, and thus does not take the form (??).

**Exercise 7.1.1.** From §6.1 we know that the general solution to (7.1.4) is

$$x(t) = \alpha \cos\left(\sqrt{\frac{k}{m}} t\right) + \beta \sin\left(\sqrt{\frac{k}{m}} t\right).$$

Show by direct computation that the quantity

$$H = \frac{1}{2}m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

is constant, thereby confirming that  $H$  is conserved.

**Exercise 7.1.2.** Suppose that  $x(t)$  is a solution to the initial value problem

$$4\frac{d^2x}{dt^2} + 9x = 0, \quad x(0) = 1, \quad x'(0) = 2.$$

Describe the trajectory of this solution in phase space. What is the maximal velocity attained by the solution? What is the maximal spatial location?

**Exercise 7.1.3.** Consider the differential equation

$$\frac{d^2x}{dt^2} = x^2.$$

- Write the equation in the form of a first-order Hamiltonian system and find the conserved energy.
- Let  $x(t)$  be the solution to the initial value problem

$$x(0) = 1, \quad x'(0) = 0.$$

What is the energy of this solution?

- Use conservation of energy to determine the trajectory that the solution takes in phase space.

**Exercise 7.1.4.** Construct a Hamiltonian system, and an initial condition, for which the corresponding solution traverses the hyperbola

$$\frac{1}{2}v^2 - \frac{1}{2}x^2 = 50.$$

How many such solutions can you find?

**Exercise 7.1.5.** Explain why the trajectories of solutions to (7.1.6) are always symmetric about the  $x$  axis.

**Exercise 7.1.6.** In this problem you study energy for a damped oscillator

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x = 0.$$

- Write this equation as a first-order system for the variables  $x$  and  $v$ .
- Let  $H = \frac{1}{2}v^2 + \frac{1}{2}x^2$ . Supposing that  $(x, v)$  is a solution to the first order system, show that  $\frac{d}{dt}E = -bv^2$ .
- Conclude that for  $b > 0$  we have  $E$  is decreasing. Give a graphical interpretation of this on the phase portrait. How can one use this fact to draw conclusions about the long-term behavior of solutions?

**Exercise 7.1.7.** In this exercise you explore a system of differential equations that have a conserved quantity that is not an energy.

- Show that  $Q = \frac{x}{y}$  is a conserved quantity for the system

$$\frac{dx}{dt} = x^2 - xy \quad \frac{dy}{dt} = xy - y^2$$

- Use the conserved quantity  $Q$  to draw the phase portrait of the system.

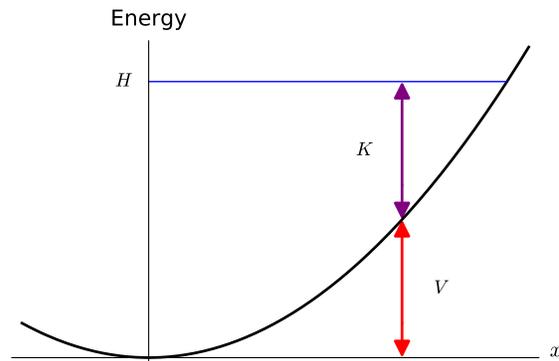
## 7.2 Energy diagrams

We now introduce a new tool for visualizing solutions to Hamiltonian systems of the form

$$\frac{dx}{dt} = v \quad m\frac{dv}{dt} = -V'(x). \quad (7.2.1)$$

An *energy diagram* is a plot of solutions where the horizontal axis is the spatial position  $x$  and the vertical axis is the energy  $H$ . Since energy is conserved, the trajectory of a solution to (7.2.1) in an energy diagram is along a horizontal line, the height of which is the energy of that solution. At each point along the trajectory, we can decompose the height, that is the energy  $H$ , into two components: the kinetic  $K = \frac{1}{2}mv^2$  and the potential energy  $V = V(x)$ . The amount of potential energy at each location is given by the potential function  $V(x)$ , and thus we can compute the amount of kinetic energy  $K$  by simply subtracting.

**Example 7.2.1.** Consider the simple harmonic oscillator, for which the potential function is  $V(x) = \frac{1}{2}kx^2$  for some constant  $k$ . In the energy diagram below, the solution traverses the blue path at height  $H$ . At any given spatial location, the total energy  $H$  is the sum of the potential part  $V$  and the kinetic part  $K$ . The amount of energy that is potential energy is indicated by the potential function  $V(x)$ , plotted in black. At each location, the vertical “gap” between the potential function and the total amount of energy indicates the amount of kinetic energy  $K$ .



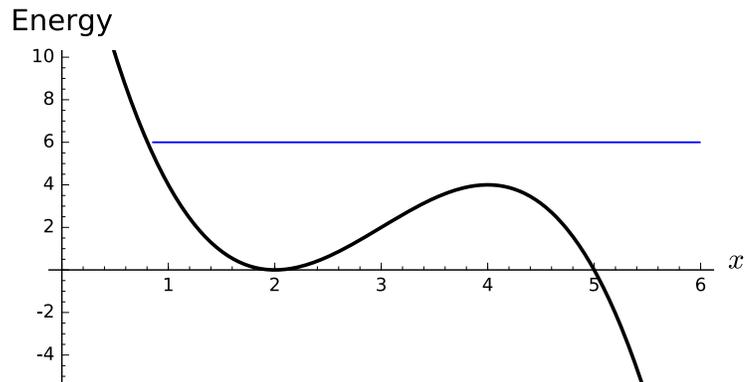
We now make a key observation: the kinetic energy  $K = \frac{1}{2}mv^2$  can never be negative. As a consequence, the trajectory of a solution in the energy diagram is always “above” the curve defined by the potential function  $V(x)$ . Thus a solution will move horizontally across the energy diagram until it reaches the graph of the potential function. At the point where the trajectory reaches this “potential curve” the kinetic energy, and thus the velocity, is zero. Hence the solution instantaneously comes to a halt. The solution, however, does not remain stationary. Consider, for example, the illustration in Example 7.2.1 above. A solution moving to the right at height  $H$  reaches the potential curve at a location where  $V'(x)$  is positive.

Thus at that moment, we deduce from the differential equation (7.2.1) that  $v' < 0$ . Thus the velocity instantaneously decreases from being zero to being negative, at which point the solution begins to move to the left. In this way we see that solutions “reflect” off the graph of the potential function.

**Example 7.2.2.** Consider the potential function  $V(x) = -x(x-2)^2(x-5)$  from Activity 7.1.1. The corresponding Hamiltonian system is

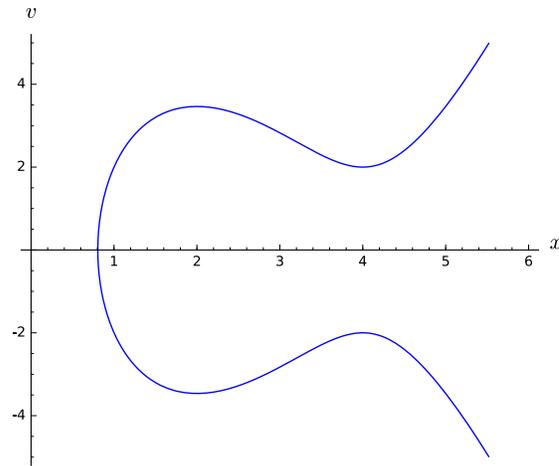
$$\frac{dx}{dt} = v \quad \frac{dv}{dt} = 3(x-2)(x-4). \quad (7.2.2)$$

The energy diagram for this potential, together with the trajectory of a solution having energy  $H = 6$  is the following:

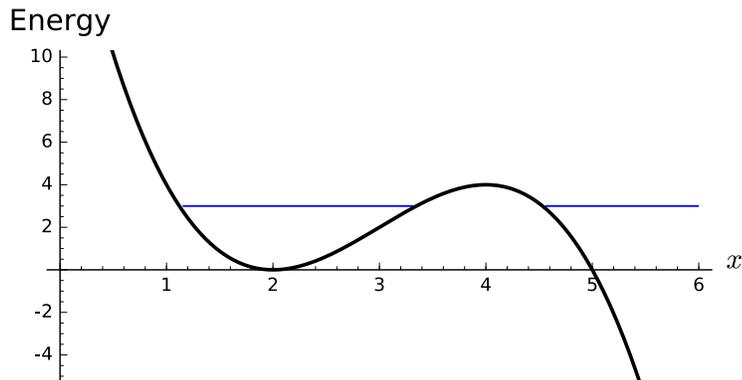


The solution begins at  $x = 6$  and negative  $v$ . It moves to the left along the line  $H = 6$  until the trajectory encounters the potential curve at  $x \approx 0.8$ . The solution then reflects off the potential wall and begins to move to the right, still along  $H = 6$ , continuing for all future times.

We can further understand this solution by plotting the trajectory in the phase plane. Initially, when  $x = 6$ , we see that the kinetic energy is large and thus  $v$  is large in magnitude; since the solution is moving to the left,  $v$  is negative. As the solution passes over the “bump” at  $x = 4$ , the magnitude of  $v$  decreases, but then increases again as the solution passes through  $x = 2$ . Then  $v$  decreases to zero as the solution approaches the potential curve. Following the reflection off the potential curve, the velocity  $v$  is positive and growing, achieving a local maximum at  $x = 2$  and then decreases as  $x$  approaches 4. Subsequently,  $v$  increases without bound. From this we can deduce that the trajectory of this solution in the phase plane is the following:



**Activity 7.2.1.** Consider again the potential  $V(x) = -x(x - 2)^2(x - 5)$ . Describe the behavior of two different solutions to (7.2.2), both having with energy  $H = 3$ , that appear in the following energy diagram.



Draw the corresponding trajectories in the phase plane.

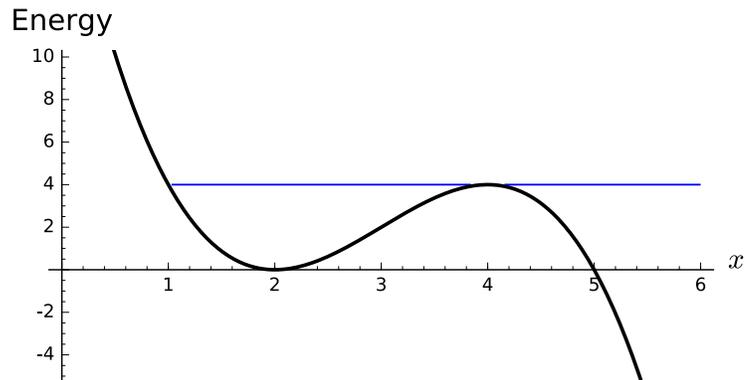
One of the solutions with energy  $H = 3$  is a **bound solution** and one is an **unbound solution**. Which is which? Use this example to develop a more general definition of these terms.

In the previous activity and example, we considered solutions to the system (7.2.2) whose trajectory in the energy diagram either passed well over the “bump” at  $x = 4$  or were not high enough to reach close to the top of that bump. However, it is possible that a solution have precisely the right energy to approach the potential curve at a local maximum of the potential function. We now investigate how solutions behave in such a situation.

Suppose that we have a potential function  $V(x)$  having a local maximum at  $x = x_*$ . This means that  $V'(x_*) = 0$ . Examining the differential equation (7.2.1), we see that  $(x, v) = (x_*, 0)$  is an equilibrium solution. In the energy diagram, this solution simply “sits” at the point  $(x, H) = (x_*, V(x_*))$ .

We now consider the situation where there is another solution  $x, v$  to (7.2.1) that has energy  $H = V(x_*)$  and whose trajectory in the energy diagram is approaching  $(x_*, V(x_*))$ . Since we cannot have two different solutions to the equation (7.2.1) meet, we conclude that this second solution  $x, v$  approaches the equilibrium point asymptotically, but never reaches it.

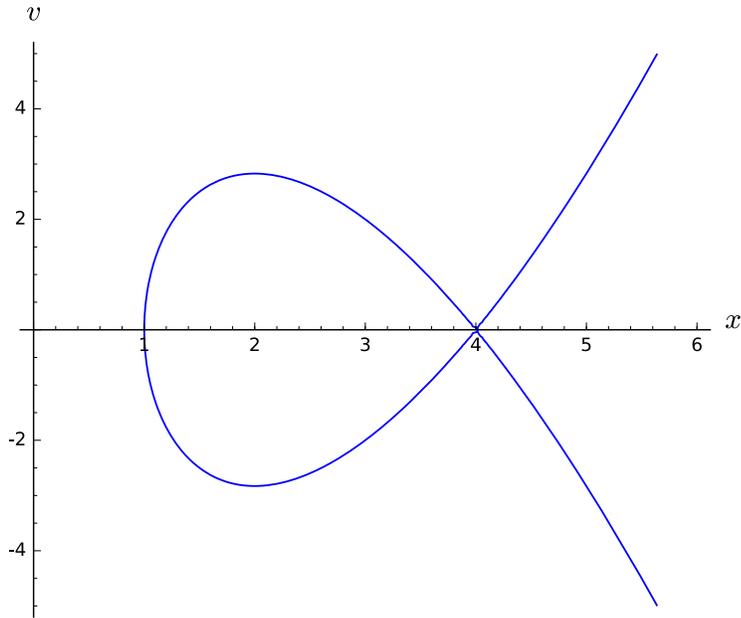
**Example 7.2.3.** Consider once more the potential function  $V(x) = -x(x - 2)^2(x - 5)$ . Solutions with energy  $H = 4$  appear in the energy diagram as follows:



Those solutions that begin to the left of  $x_* = 4$  with negative velocity move to the left until they reflect off the potential curve; then they move to the right, asymptotically approaching  $x_* = 4$ . Those solutions that initially have  $x < 4$  but begin with positive velocity simply asymptote directly to  $x_* = 4$ .

Solutions to the right of  $x_* = 4$  tend towards the equilibrium if their initial velocity is negative, but away from it if their initial velocity is positive.

In the phase diagram, these trajectories are as follows:



We now investigate the equilibrium points for (7.2.1) more systematically. Our first observation is that *all* of the equilibrium points take the form  $(x, v) = (x_*, 0)$ , where  $x_*$  is a root of  $V'(x)$ . In other words, the equilibrium points of a Hamiltonian system are precisely the critical points of the potential function.

**Activity 7.2.2.** Suppose  $(x, v) = (0, x_*)$  is an equilibrium point for (??). Show that the linearization of (7.2.1) at  $(0, x_*)$  is given by

$$\frac{d}{dt} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_*) & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}.$$

Using the results of this activity, we see that the eigenvalues of the linearized system are

$$\lambda = \pm \sqrt{-V''(x_*)}.$$

Thus:

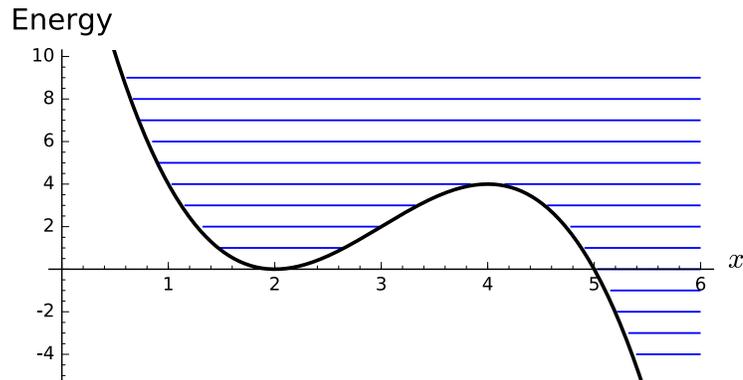
- If  $V''(x_*) > 0$ , then  $(x_*, 0)$  is a center-type equilibrium.
- If  $V''(x_*) < 0$ , then  $(x_*, 0)$  is a saddle-type equilibrium.

Of course, if  $V''(x_*) = 0$ , then we do not learn anything about the stability type of the equilibrium from the linearization. However, it is still

the case that if  $x_*$  is a local minimum of  $V$  then  $(x_*, 0)$  is a center and if  $x_*$  is a local maximum of  $V$  then  $(x_*, 0)$  is a saddle. If  $x_*$  is a critical point of  $V$  but is neither a local minimum nor local maximum, then one obtains a **degenerate equilibrium**; see Exercise 7.2.3.

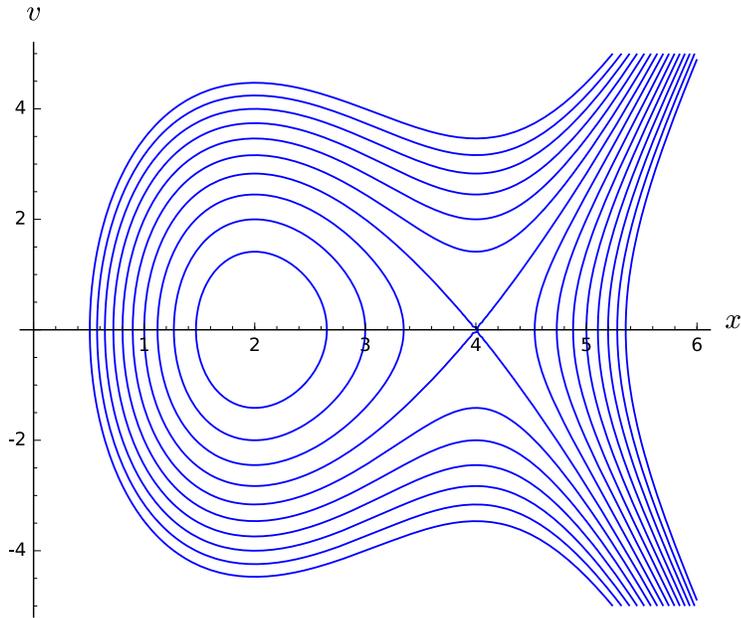
The result of this discussion is that we can entirely understand the solutions to a Hamiltonian system by examining the shape of the potential function. Critical points of the potential function are equilibrium points, and are either saddle-, center-, or degenerate-type. By considering solutions of various energies, we can deduce which initial conditions give rise to bound solutions, and which give rise to unbound solutions. Finally, by tracing each solution through the energy diagram, we can easily deduce the shape of the corresponding solution in the phase plane.

**Example 7.2.4.** Consider one final time the potential function  $V(x) = -x(x-2)^2(x-5)$ . Here we plot the energy diagram with a variety of solution trajectories having different energy levels:



Notice that we have a center-type equilibrium solution at  $x_* = 2$  and a saddle-type equilibrium solution at  $x_* = 4$ . Solutions with energy  $H < 4$  and having  $x(0) < 4$  will be bound solutions; these solutions oscillate about the equilibrium  $x_* = 2$ . All solutions with  $H > 4$  or with  $H < 4$  but  $x(0) > 4$  have  $x \rightarrow \infty$  as  $t \rightarrow \pm\infty$ . Finally, solutions with  $H = 4$  will tend to the equilibrium at  $x_* = 4$ , unless we have both  $x(0) > 4$  and  $v(0) > 0$ , in which case  $x \rightarrow \infty$  as  $t \rightarrow \infty$ .

Using this information, we can easily draw the corresponding trajectories in the phase plane:



**Exercise 7.2.1.** Suppose we have the forced oscillator

$$\frac{d^2x}{dt^2} + 9x = 10$$

1. Write this as a first order system.
2. Show that  $H = \frac{1}{2}v^2 + \frac{9}{2}x^2 - 10x$  is a conserved quantity.
3. Draw the energy diagram for the equation.
4. Draw the phase portrait for the equation.
5. Discuss the long-term behavior of solutions to the system, based on your diagram & portrait.
6. Find the general solution to the differential equation. Does the behavior match what the pictures predict?

**Exercise 7.2.2.** Consider the differential equation

$$\frac{d^2x}{dt^2} = x^3 - x$$

1. Write the equation as a first-order system.

2. What is the conserved quantity; what is the potential function  $V$ ?
3. Carefully draw the energy diagram. Be sure to include several typical trajectories.
4. Carefully draw the phase portrait. Include the paths corresponding to the typical trajectories you put on the energy diagram.
5. Analyze the system by making some statements of the sort: *Suppose initially \_\_\_\_\_, then the solution...*

**Exercise 7.2.3.** Here you analyze the prototypical case of a degenerate critical point of the potential function.

- Work with the potential function  $V(x) = x^3$ .
- Write down the corresponding Hamiltonian system.
- Linearize the system at the equilibrium point  $(0, 0)$ .
- Compute the eigenvalues of the matrix for the linearized system, and verify that this situation is indeed “degenerate” in the sense that the linearized system doesn’t have eigensolutions.

**Exercise 7.2.4.** In this exercise you explore the nullclines of Hamiltonian systems.

1. First consider the generic Hamiltonian system (7.2.1). Describe the nullclines of this system; your description will likely involve the potential function. How can you deduce what the nullclines will be from the plot of the potential function  $V$ ?
2. Find the nullclines of the system in Exercise 7.2.2.

### 7.3 Examples of Hamiltonian systems

In this section we practice analyzing Hamiltonian systems.

**Activity 7.3.1.** Consider the equation

$$\frac{d^2x}{dt^2} = 3(1 - x^2).$$

Write the equation as a first order system and find the corresponding potential function. Use the potential function to analyze solutions to the equation by

- *drawing both the energy and phase diagrams,*
- *identifying the equilibrium points and nullclines,*
- *determining the bound and unbound solutions, and*
- *determining the “critical energies” at which the behavior of solutions changes.*

**Activity 7.3.2.** *Consider the equation*

$$\frac{d^2x}{dt^2} = e^x.$$

*Write the equation as a first order system and find the corresponding potential function. Use the potential function to analyze solutions to the equation by*

- *drawing both the energy and phase diagrams,*
- *identifying the equilibrium points and nullclines,*
- *determining the bound and unbound solutions, and*
- *determining the “critical energies” at which the behavior of solutions changes.*

**Activity 7.3.3.** *Consider the potential function  $V(x) = (1 - x)^2(1 + x)^2$ . Write down the corresponding Hamiltonian system. Use the potential function to analyze solutions to the system by*

- *drawing both the energy and phase diagrams,*
- *identifying the equilibrium points and nullclines,*
- *determining the bound and unbound solutions, and*
- *determining the “critical energies” at which the behavior of solutions changes.*

**Activity 7.3.4.** *Consider the potential function  $V(x) = \frac{1}{1+x^2}$ . Write down the corresponding Hamiltonian system. Use the potential function to analyze solutions to the system by*

- *drawing both the energy and phase diagrams,*

- *identifying the equilibrium points and nullclines,*
- *determining the bound and unbound solutions, and*
- *determining the “critical energies” at which the behavior of solutions changes.*

**Activity 7.3.5.** Consider the potential function  $V(x) = -xe^{-x}$ . Write down the corresponding Hamiltonian system. Use the potential function to analyze solutions to the system by

- *drawing both the energy and phase diagrams,*
- *identifying the equilibrium points and nullclines,*
- *determining the bound and unbound solutions, and*
- *determining the “critical energies” at which the behavior of solutions changes.*

**Exercise 7.3.1.** Perform a full Hamiltonian analysis of differential equation

$$\frac{d^2x}{dt^2} = \frac{1}{(x+1)^2}$$

**Exercise 7.3.2.** Perform a full Hamiltonian analysis of differential equation

$$\frac{d^2x}{dt^2} = e^{-x} - 1.$$

**Exercise 7.3.3.** Perform a full Hamiltonian analysis of the differential equation arising from the potential function  $V(x) = x^2 - 2x + 1$

**Exercise 7.3.4.** Perform a full Hamiltonian analysis of the differential equation arising from the potential function  $V(x) = \frac{1}{x}$

**Exercise 7.3.5.** Perform a full Hamiltonian analysis of the differential equation arising from the potential function  $V(x) = \frac{1}{x} + x$

**Exercise 7.3.6.** Perform a full Hamiltonian analysis of the differential equation arising from the potential function  $V(x) = \frac{-1}{1+x^2}$