Chapter 6

Oscillations and resonance

6.1 Modeling oscillations

In this chapter we study equations that model one-dimensional oscillations. Let's begin by with some formulas borrowed from physics. At this stage, we won't concern ourselves with how the physicists obtained these formulas, though that is a very interesting subject. Rather, we accept that the physicists have good reasons for believing that the formulas are relevant and proceed to mathematically analyze the consequences of accepting the formulas.

Our first formula comes from Newton, who asserted that objects are subject to "forces" and that the change in momentum of an object is equal to the total force acting upon the object. To convert this in to a formula, we first suppose that we have an object of mass m that is only able to move in one direction, which we call the y-direction. We describe the movement of the object by the function $y(t)$, which tells us the location of the "center of mass" of the object at each time t. The ***momentum*** of the object is defined to be

$$p = m\frac{dy}{dt},$$

and is thus a function as well. (Technically, momentum is a vector. However, since we are only considering motion in one direction the momentum vector has only one component. Thus we are able to describe the momentum by a single function.) We can express Newton's assertion that the change in momentum is equal to the force by the formula

$$F = \frac{dp}{dt}, \tag{6.1.1}$$

which is known as "Newton's Second Law." It is common to rewrite (6.1.1) by computing

$$\frac{dp}{dt} = m\frac{d^2y}{dt^2}$$

and labeling the acceleration $\frac{d^2y}{dt^2}$ with the letter a; thus (6.1.1) becomes the more famous formula

$$F = ma.$$

However, since this is a differential equations course it is more convenient for us to keep the derivatives explicitly present. Thus we either write

$$F = \frac{dp}{dt} \qquad \text{and} \qquad p = m\frac{dy}{dt}$$

or we write

$$F = m\frac{d^2y}{dt^2}.$$

The second formula from physics that we need is due to Hooke (who, incidentally, corresponded with Newton engaged in a priority dispute with him concerning the inverse square law of gravitation). Suppose our object of mass m is attached to a spring in such a way that the object will remain at rest if placed at $y = 0$, but that the object will feel a force from the spring if displaced from location $y = 0$. Hooke's Law is the postulation that the force of the spring upon the mass is proportional to the displacement. Mathematically, we express this assertion as

$$F_{\text{spring}} = -ky,$$

where $k > 0$ is known as the "spring constant."

Together, Newton's Second Law and Hooke's Law combine to give us the differential equation

$$m\frac{d^2y}{dt^2} = -ky,$$

which is known as the ***simple harmonic oscillator***.

The simple harmonic oscillator is a basic model of oscillations. More complicated models can be constructed by including terms that account for external forces and for friction. The most class of oscillator models we consider in this course take the form

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f, \tag{6.1.2}$$

where the $-b\frac{dy}{dt}$ is a simple model for ***frictional damping*** and f describes any external ***forcing*** present. Here b is a constant, but f might be a function of t.

Example 6.1.1. *Near the surface of the earth, the force of gravity upon an object of mass m is given by $f = -mg$, where $g \approx 9.8\, m/s^2$ and the negative sign is due to the fact that the force is downward. Thus the height y of an oscillating mass hanging from a spring can be modeled with the differential equation*

$$m\frac{d^2y}{dt^2} = -ky - mg.$$

There are two "standard" ways to write the generic oscillator equation (6.1.2). The first standard way to write it is to put all the y terms on the left, leaving the forcing on the right:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f. \tag{6.1.3}$$

Equations of the form (6.1.3) are called ***constant-coefficient, second-order*** because they involve second derivatives of the unknown y and because the coefficients m, b, k are constants.

The second standard way to write (6.1.2) is as a first-order system. We can do this by introducing a new variable, the velocity

$$v = \frac{dy}{dt}. \tag{6.1.4}$$

If we replace every instance of $\frac{dy}{dt}$ with v, then (6.1.2) becomes

$$m\frac{dv}{dt} = -ky - bv + f. \tag{6.1.5}$$

Thus (6.1.2) can be written as the system

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v + \frac{f}{m}. \tag{6.1.6}$$

As an aside, we remark that in classical mechanics it is common to work in terms of the momentum p rather than the velocity v. In this case, the system of equations becomes

$$\frac{dy}{dt} = \frac{1}{m}p, \qquad \frac{dp}{dt} = -ky - \frac{b}{m}p + f.$$

Of course, the two are mathematically equivalent…and in fact we could easily re-do the rest of this section using the p variable. But we'll stick with the variable v in these notes.

We will frequently go back and forth between the second-order formulation (6.1.3) and the first order formulation (6.1.6). We illustrate this with the following example.

Example 6.1.2. *The simple harmonic oscillator can be written as either*

$$m\frac{d^2y}{dt^2} + ky = 0$$

or as

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -\frac{k}{m}y.$$

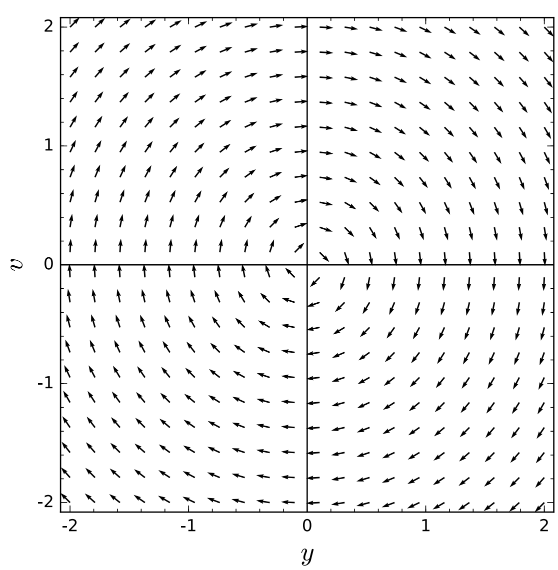
The first order formulation we can write as

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

We can easily solve the first-order system using eigenstuff. First, we compute the eigenvalues to be

$$\lambda_{\pm} = \pm i\sqrt{\frac{k}{m}}.$$

Thus there is a center equilibrium at $(y, v) = (0, 0)$*. It is easy to verify that the direction of rotation is clockwise; consequently the phase plane picture looks something like this:*



We can easily read off the phase plane that the function $y(t)$ is periodic. Furthermore, when y is large, its derivative $v = \frac{dy}{dt}$ is small and vice-versa. Thus we learn something about solution to the second-order equation by looking at the phase portrait for the first-order system!

We now use the first-order system to construct solutions to the second-order equation. Focusing on λ_+ we choose eigenvector

$$\begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix}.$$

Thus we obtain the complex eigensolution

$$Y_+(t) = e^{i\sqrt{\frac{k}{m}}\,t}\begin{pmatrix} 1 \\ i\sqrt{\frac{k}{m}} \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} + i\begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

From this we obtain two independent real solutions

$$\begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

Thus the general solution to the first order formulation of the simple harmonic oscillator is

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \alpha\begin{pmatrix} \cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix} + \beta\begin{pmatrix} \sin\left(\sqrt{\frac{k}{m}}\,t\right) \\ \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) \end{pmatrix}.$$

We can translate these results back to second-order equation. Reading off the first components of the vector we see that the general solution for y is given by

$$y(t) = \alpha\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \beta\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

Actually, we can learn one more thing about the second order equation for the simple harmonic oscillator by looking at the first-order system. We know that an appropriate initial condition for the first-order system takes the form

$$\begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

for some constants y_0 and v_0. In fact, we can easily see that the solution to the corresponding initial value problem is obtained by choosing $\alpha = y_0$ and $\beta = v_0\sqrt{\frac{m}{k}}$. From this we learn that the appropriate initial conditions for the second-order equation take the form

$$y(0) = y_0 \qquad \textit{and} \qquad \frac{dy}{dt}(0) = v_0,$$

and that the solution to the corresponding initial value problem is given by

$$y(t) = y_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

In general, given a second order equation of the form (6.1.3) we can form an associated first-order system of the form (6.1.6). The previous example illustrates how to use the phase diagram of the first order system in order to describe solutions to the second-order equation. We also see that the appropriate initial conditions for a second order equation involve specifying both the initial value and initial value of the first derivative of the unknown. Physically this means specifying the initial position and the initial velocity.

Example 6.1.3. *Consider the second-order equation*

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0.$$

This equation is equivalent to the first-order system

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

The characteristic equation for the matrix in this system is

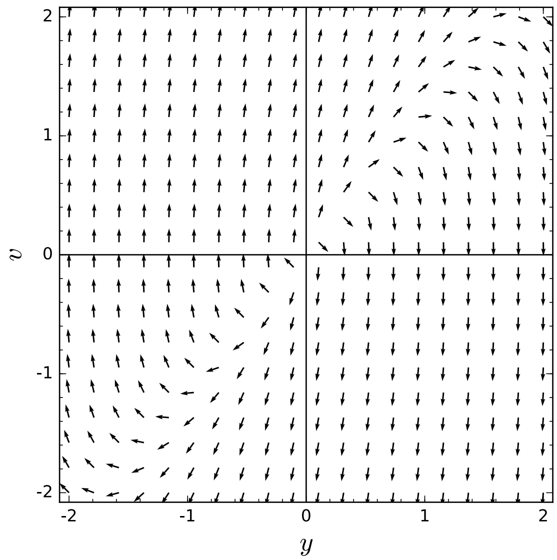
$$\lambda^2 - 5\lambda + 6 = 0$$

and thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

We compute the corresponding eigenvectors to be

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Thus the phase portrait for the system looks like



From this we see that if $v_0 > 2y_0$ *then we have* $v, y \to \infty$ *as* $t \to \infty$*, while if* $v_0 < y_0$ *then* $v, y \to -\infty$*. If* $v_0 = 2y_0$*, then the* $v, y \to \pm\infty$ *with the sign equal to the sign of* y_0*.*

Since the general solution to the first order system is

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

we may deduce that the general solution to the original second order equation is

$$y(t) = \alpha e^{2t} + \beta e^{3t}.$$

This last example suggests a close connection the general solution to the second order equation and the eigenvalues of the matrix associated to the associated first order system. This connection is discussed in more detail in the next section.

Activity 6.1.1. *Write the second order equation*

$$3\frac{d^2y}{dt^2} + 13\frac{dy}{dt} - 10y = 0$$

as a first order system. Use the first order system to determine how solutions to the original second order equation behave.

Exercise 6.1.1. Re-write the following second order equations as the first order system. Then use `Sage` to generate the phase portraits of the oscillators. Use the phase portrait to discuss the behavior of a 'typical' solution $y(t)$.

1. $\dfrac{d^2y}{dt^2} + 4y = 0;$
2. $\dfrac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3;$
3. $\dfrac{d^2y}{dt^2} + 3y^2 = 3.$

Exercise 6.1.2. In this problem you study the equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 4y = 0,$$

for various values of $b \geq 0$.

1. Write this equation as a first-order system.
2. Use the following `Sage` code to explore the phase portrait for various values of b:

```
var('x,y')
@ interact
def _(b = (0,(0,5)) ):
    Field = [y,-4*x-b*y]
    Fieldplot = plot_vector_field(vector(Field)/vector(
        Field).norm(),(x,-2,2),(y,-2,2))
    Fieldplot.show(figsize=[5,5],axes_labels=['$y$','$v$'
        ])
```

For what values of b do solutions oscillate? Can you interpret your results "physically"?

Exercise 6.1.3. Consider the simple harmonic oscillator (SHO) equation

$$m\frac{d^2y}{dt^2} + ky = 0.$$

1. Let $\omega = \sqrt{k/m}$ and show that the first-order system for SHO can be written
$$\frac{d}{dt}Y = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} Y.$$

2. Show that the corresponding propagator matrix $P(t)$ is

$$P(t) = \begin{pmatrix} \cos(t\omega) & \frac{\sin(t\omega)}{\omega} \\ -\omega\sin(t\omega) & \cos(t\omega) \end{pmatrix}$$

3. Use $P(t)$ to solve the IVP

$$\frac{d^2y}{dt^2} + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

4. Let $Y(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$ be the solution to the IVP in part (3). Show that the path $(y(t), v(t))$ travels along the ellipse

$$y^2 + \frac{v^2}{4} = 1.$$

5. Do other solutions to the equation in part (3) also travel along ellipses?

6.2 Constant coefficient homogeneous equations

Our plan for studying oscillator-type equations is the following. First, in this section, we study equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{6.2.1}$$

where a, b, c are constants. (We assume that $a \neq 0$ because otherwise we have a first-order equation.) Notice that the generic oscillator model (6.1.3) takes this form if the forcing is set to zero. In subsequent sections we introduce forcing terms on the right hand side.

Equations of the form (6.2.1) are called ***constant coefficient*** because the coefficients are constant, and are called ***homogeneous*** because the right side of the equation is zero.

One way to study the equation (6.2.1) is to express it as the first order system

$$\frac{dy}{dt} = v \qquad \frac{dv}{dt} = -\frac{c}{a}y - \frac{b}{a}v.$$

This can also be written in vector-matrix form as

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$-\lambda\left(-\frac{b}{a}-\lambda\right)+\frac{c}{a}=0,$$

which simplifies to

$$a\lambda^2+b\lambda+c=0. \tag{6.2.2}$$

Suppose now that λ is a solution to (6.2.2). The corresponding eigenvector must satisfy

$$\begin{pmatrix}0 & 1\\ -\frac{c}{a} & -\frac{b}{a}\end{pmatrix}\begin{pmatrix}\bigstar\\ \clubsuit\end{pmatrix}=\lambda\begin{pmatrix}\bigstar\\ \clubsuit\end{pmatrix}.$$

In particular, we must have

$$\clubsuit=\lambda\bigstar$$

and therefore we may choose the eigenvector associated to λ to be

$$\begin{pmatrix}1\\ \lambda\end{pmatrix}.$$

The associated eigensolution to the first order system is

$$e^{\lambda t}\begin{pmatrix}1\\ \lambda\end{pmatrix}.$$

From this we conclude the following: *If λ is a solution to* (6.2.2)*, then $y(t)=e^{\lambda t}$ is a solution to* (6.2.1)*.*

In fact, we could have learned this a much easier way. If we plug the function $y(t)=e^{\lambda t}$ in to the equation (6.2.1), we obtain

$$e^{\lambda t}\left(a\lambda^2+b\lambda+c\right)=0. \tag{6.2.3}$$

Since $e^{\lambda t}$ is not the zero function, we again see that $e^{\lambda t}$ is a solution to (6.2.1) precisely when λ solves (6.2.2).

We now know how to construct "simple" solutions to (6.2.1). In order to use this to construct a general solution, we need to know that the superposition principle works for the equation (6.2.1). By direct computation, we can verify the following: *If $y_1(t)$ and $y_2(t)$ are solutions, then so is $y(t)=\alpha y_1(t)+\beta y_2(t)$ for any constants α,β*; see the exercises.

Since we now that the superposition principle works for the equation (6.2.1) we can now proceed as follows. If (6.2.2) has two real solutions λ_1 and λ_2, then we know that both

$$y_1(t)=e^{\lambda_1 t} \qquad \text{and} \qquad y_2(t)=e^{\lambda_2 t}$$

are solutions. Using the superposition principle, we find that the generic solution to (6.2.1) is

$$y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}.$$

If, however, the equation (6.2.2) has complex solutions $\lambda_\pm = a \pm bi$ then we can use the superposition principle to conclude that

$$y_1(t) = e^{at}\cos(bt) \qquad \text{and} \qquad y_2(t) = e^{at}\sin(bt)$$

are solutions. From this we deduce that a general solution to (6.2.1) is

$$y(t) = \alpha e^{at}\cos(bt) + \beta e^{at}\sin(bt);$$

see the exercises for details.

Example 6.2.1. *Let's find the general solution to the equation*

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

We see that $e^{\lambda t}$ is a solution when

$$0 = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3).$$

Thus both e^{-2t} and e^{-3t} are solutions, and a general solution is

$$y(t) = \alpha e^{-2t} + \beta e^{-3t}.$$

Notice that $y(t)$ decays to zero without any oscillation as $t \to \infty$.

While we can easily plot typical solutions $y(t)$ on the t–y axis, we can also plot solutions in the y–v phase plane. Setting $v = \frac{dy}{dt}$ we have

$$\begin{pmatrix} y \\ v \end{pmatrix} = \alpha e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

From this we see conclude that the phase portrait has a sink-type equilibrium at $(y, v) = (0, 0)$.

Example 6.2.2. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 6y = 0.$$

We see that $e^{\lambda t}$ is a solution when

$$0 = \lambda^2 + \lambda + 6,$$

which occurs when

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{23}}{2}\, i.$$

Thus the general solution is

$$y(t) = \alpha e^{-t/2} \cos\left(\frac{\sqrt{23}}{2} t\right) + \beta e^{-t/2} \sin\left(\frac{\sqrt{23}}{2} t\right).$$

Notice that these solutions oscillate with exponentially decaying amplitude.

We now want to plot the solution in phase space. Since the eigenvalues λ *are complex with negative real part, the equilibrium at zero is a spiral sink. In order to determine the direction, we observe that* $v > 0$ *means that* y *is increasing. Thus the spiral is clockwise.*

If we wanted an exact formula for the trajectories in phase space, we can compute $v = \frac{dy}{dt}$*, from which we deduce that*

$$\begin{pmatrix} y \\ v \end{pmatrix} = \alpha \left\{ e^{-t/2} \cos\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 0 \\ -\sqrt{23}/2 \end{pmatrix} \right\}$$
$$+ \beta \left\{ e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{-t/2} \sin\left(\tfrac{\sqrt{23}}{2} t\right) \begin{pmatrix} 0 \\ \sqrt{23}/2 \end{pmatrix} \right\}.$$

We conclude this section with some remarks about the superposition principle and the concept of linearity. Previously, we defined a first order system to be linear if it took the form

$$\frac{d}{dt} Y = F(Y)$$

for some linear function F. One consequence of this was the superposition principle: if $Y_1(t)$ and $Y_2(t)$ are solutions, then so is $Y(t) = \alpha Y_1(t) + \beta Y_2(t)$ for any constants α, β.

We know that we can write (6.2.1) as a first order system. Thus one way to define what it means for a second order equation to be linear would be that the associated first order system is linear. This is a fine definition, but it is not the standard one. Rather, we make use of the superposition principle directly and say that (6.2.1) is defined to be ***linear*** because it has the property that if $y_1(t)$ and $y_2(t)$ are solutions, then so is $y(t) = \alpha y_1(t) + \beta y_2(t)$ for any constants α, β.

Exercise 6.2.1. Verify that the superposition principle works for the equation (6.2.1) as follows. Assume that $y_1(t)$ and $y_2(t)$ are solutions. Show by direct computation that this implies that $y(t) = \alpha y_1(t) + \beta y_2(t)$ is a solution for any constants α, β.

Exercise 6.2.2. Suppose $e^{\lambda t}$ is a solution to (6.2.1) with $\lambda = a + bi$. Show how to use this in order to conclude that

$$y_1(t) = e^{at}\cos(bt) \qquad \text{and} \qquad y_2(t) = e^{at}\sin(bt)$$

are solutions.

Exercise 6.2.3. Consider the second order differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

1. Find the general solution to this differential equation.

2. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Exercise 6.2.4. Show that the superposition principle also holds for ***non-constant coefficient homogeneous linear equations***, which take the form

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = 0$$

for functions a, b, c.

Exercise 6.2.5. Consider the differential equation

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0.$$

1. Find those values of α for which the function $y(t) = t^\alpha$ solves the differential equation.

2. Use the superposition principle from Exercise 6.2.4 to solve the IVP:

$$t^2\frac{d^2y}{dt^2} - 3t\frac{dy}{dt} + 3y = 0, \qquad y(1) = 2, \quad y'(1) = 4.$$

Exercise 6.2.6. Find the general solution of the following equations:

1. $\dfrac{d^2y}{dt^2} + \omega^2 y = 0;$

2. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0;$

3. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 0$;

4. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0$;

5. $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$.

6.3 Damped oscillators

(the contents of this section have been moved to the writing assignment)

6.4 Oscillators with forcing

In the previous sections we developed a good understanding of solutions to homogeneous equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{6.4.1}$$

Our goal now is to understand what happens when we introduce forcing. Thus we study equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f, \tag{6.4.2}$$

where f is some function of t, and where a, b, c are constants.

In order to handle forcing, we introduce the following "generalized superposition principle."

Theorem (Generalized superposition principle)**.** *Suppose $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous equation* (6.4.1) *and that $y_p(t)$ is a solution to the inhomogeneous equation* (6.4.2)*. Then*

$$y(t) = \alpha y_1(t) + \beta y_2(t) + y_p(t)$$

is a solution to (6.4.2) *for any constants α and β.*

The generalized superposition principle can be physically interpreted as follows: Suppose we have an inhomogeneous equation of the form (6.4.2). Then the general solution

$$y_h(t) = \alpha y_1(t) + \beta y_2(t) \tag{6.4.3}$$

to the associated homogeneous equation (6.4.1) is the "natural response" of the system, in the absence of any external forcing. The function $y_h(t)$ is often called the ***homogeneous solution***. The function $y_p(t)$ is an additional contribution to $y(t)$ that represents the "response" of the system to the external forcing. The function $y_p(t)$ is usually called a ***particular solution***.

The generalized superposition principle suggests an approach for finding the general solution to equations of the form (6.4.2):

1. Find the general solution $y_h(t)$ to the homogeneous equation,
2. Find a particular solution $y_p(t)$ to the inhomogeneous equation,
3. Construct the general solution $y(t) = y_p(t) + y_h(t)$ to the inhomogeneous equation.

Since homogeneous equations are well understood, the general superposition principle means that if we can find one solution to an inhomogeneous equation, then we can easily find them all!

However, we still have to find one solution to (6.4.2). Unfortunately, the most efficient method that mathematicians have been able to come up with thus far is... educated guess and check. The guiding principle is this:

> *Look for a particular solution $y_p(t)$ that is the same "type" of function that $f(t)$ is.*

If you find this unsatisfying, you have good company. There is general theory out there about constructing particular solutions. But the educated guess-and-check method is so much more efficient for the simple cases we'll cover in this class that it simply isn't worth it to get out the fancy theory at this point.

The following list of examples demonstrates how the educated guess and check method works.

Example 6.4.1. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$, and thus the eigenvalues are $\lambda = \pm 2i$. This implies that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f is constant, we guess that the particular solution takes the form $y_p(t) = a$ for some constant a. Plugging this in to the original equation yields

$$0 + 4a = 7.$$

Thus we obtain a solution $y_p(t) = 7/4$.

Assembling the pieces, we see that the general solution is

$$y(t) = \alpha \cos(2t) + \beta \sin(2t) + \frac{7}{4}.$$

Example 6.4.2. *Let's find the general solution to*

$$\frac{d^2y}{dt^2} + 4y = 7t.$$

First we consider the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

The characteristic equation is $\lambda^2 + 4 = 0$, and thus the eigenvalues are $\lambda = \pm 2i$. This implies that the homogeneous solution is

$$y_h(t) = \alpha \cos(2t) + \beta \sin(2t).$$

We now try to guess that a particular solution. Since the forcing function f is a linear function, we guess that the particular solution takes the form $y_p(t) = a + bt$ for some constants a, b. Plugging this in to the original equation yields

$$0 + 4(a + bt) = 7t.$$

We rearrange this to

$$4a + (4b - 7)t = 0.$$

In this last equation, we have equality in the sense of functions. Thus we must have $a = 0$ and $b = 7/4$. Thus the particular solution is

$$y_p(t) = \frac{7}{4}t$$

and the general solution is

$$y(t) = \alpha \cos(2t) + \beta \sin(2t) + \frac{7}{4}t.$$

Example 6.4.3. *Let's analyze the solutions to the equation*

$$\frac{d^2y}{dt^2} + 4y = e^{2t}.$$

First we address the homogeneous equation

$$\frac{d^2y}{dt^2} + 4y = 0.$$

From the previous examples, we know that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We now look for a particular solution. Since the forcing function is exponential with growth rate 2, we look for a particular solution of the form

$$y_p(t) = ae^{2t}$$

where a is some constant. Plugging this in to the differential equation yields

$$4ae^{2t} + 4ae^{2t} = e^{2t}.$$

It is easy to see that choosing $a = 1/8$ leads to a particular solution

$$y_p(t) = \frac{1}{8}e^{2t}$$

and thus the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) + \frac{1}{8}e^{2t}.$$

Notice that as $t \to \infty$, we have $y_p(t) \gg y_h(t)$ and thus the forced response dominates behavior far in the future. As $t \to -\infty$ we have $y_p(t) \ll y_h(t)$ and thus the natural response dominates far in the past.

Activity 6.4.1. *Find the general solution to the differential equation*

$$\frac{d^2y}{dt^2} + 4y = e^{-2t}.$$

In what regime is y_h dominant? In what regime is y_p dominant?

Finally, we give an example in which we solve an initial value problem.

Example 6.4.4. *Let's solve the initial value problem*

$$\frac{d^2y}{dt^2} + 16y = 2t + 1, \qquad y(0) = 2, \quad y'(0) = 0.$$

To accomplish this, we first find the general solution.

The homogeneous equation is

$$\frac{d^2y}{dt^2} + 16y = 0,$$

which has characteristic equation $\lambda^2 + 16 = 0$. *Thus the eigenvalues are* $\lambda = \pm 4i$ *and the homogeneous solution is*

$$y_h(t) = \alpha\cos(4t) + \beta\sin(4t).$$

Since the forcing function is linear, we look for a particular solution of the form $y_p(t) = a + bt$. *Plugging this in to the original equation yields*

$$16(a + bt) = 2t + 1.$$

This is satisfied if we choose $a = 1/16$ *and* $b = 1/8$. *Thus the particular solution is*

$$y_p(t) = \frac{1}{16} + \frac{1}{8}t$$

and the general solution is

$$y(t) = \alpha\cos(4t) + \beta\sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

We now impose the initial conditions. The condition that $y(0) = 2$ *becomes*

$$2 = \alpha + \frac{1}{16}.$$

Computing

$$y'(t) = -4\alpha\sin(4t) + 4\beta\cos(4t) + \frac{1}{8}$$

we see that the condition $y'(0) = 0$ *becomes*

$$0 = 4\beta + \frac{1}{8}.$$

Thus $\alpha = 31/16$ *and* $\beta = -1/32$, *which implies that the solution to the initial value problem is*

$$y(t) = \frac{31}{16}\cos(4t) - \frac{1}{32}\sin(4t) + \frac{1}{16} + \frac{1}{8}t.$$

Exercise 6.4.1. Consider the non-homogeneous equation:

$$\frac{d^2y}{dt^2} + 9y = 9.$$

This equation arises from studying a frictionless oscillator with constant forcing.

1. Find a particular solution of this equation.
2. Find the homogeneous solution.
3. Based on the above find the general solution of the equation.
4. Solve the IVP
$$\frac{d^2y}{dt^2} + 9y = 9, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3.$$
5. Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.
6. How would you in words describe the effect the forcing has on the oscillator?

Exercise 6.4.2. Repeat Problem 6.4.1 for the differential equation $\frac{d^2y}{dt^2} + 9y = 10e^{-t}$ and IVP

$$\begin{cases} \frac{d^2y}{dt^2} + 9y = 10e^{-t}, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = -7. \end{cases}$$

Exercise 6.4.3. Repeat Problem 6.4.1 for the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1$ and the IVP

$$\begin{cases} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1, \\ y(0) = 0, \quad \frac{dy}{dt}(0) = 0. \end{cases}$$

Note: this equation represents an oscillator with friction.

Exercise 6.4.4. Repeat Problem 6.4.1 for the equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t$ and the IVP

$$\begin{cases} \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1 + 3e^t, \\ y(0) = 2, \quad \frac{dy}{dt}(0) = 1. \end{cases}$$

Exercise 6.4.5. Solve the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 16y = e^{-\frac{t}{10}} \\ y(0) = \dfrac{110}{1601}, \quad y'(0) = \dfrac{-10}{1601} \end{cases}$$

Plot the solution and describe its characteristic features.

Exercise 6.4.6. Find general solutions of the following differential equations.

1. $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 4e^{-2t}$;
2. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 1 + e^{-t}$.

Exercise 6.4.7. Find the general solution of the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 2t + 1.$$

Exercise 6.4.8. Devise a recipe for finding a forced response of the oscillator with polynomial forcing $f(t) = a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0$.

6.5 Oscillators with trigonometric forcing

In this section we address oscillators with trigonometric forcing. If there is no damping present in the oscillator, then we can proceed essentially as in the previous section.

Example 6.5.1. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(3t).$$

We easily see that the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

Since the forcing function is a cosine function with frequency 3*, we look for a particular solution of the form* $y_p(t) = a\cos(3t)$*. Plugging this in to the equation yields*

$$-9a\cos(3t) + 4a\cos(3t) = \cos(3t).$$

From this we deduce that $a = -1/5$ *and thus we have*

$$y_p(t) = -\frac{1}{5}\cos(3t).$$

The general solution is therefore

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{5}\cos(3t).$$

In order to understand this solution, it is helpful to make use of the trigonometric identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to write

$$\cos(3t) = \cos(2t)\cos(t) - \sin(2t)\sin(t).$$

Thus the general solution can be written

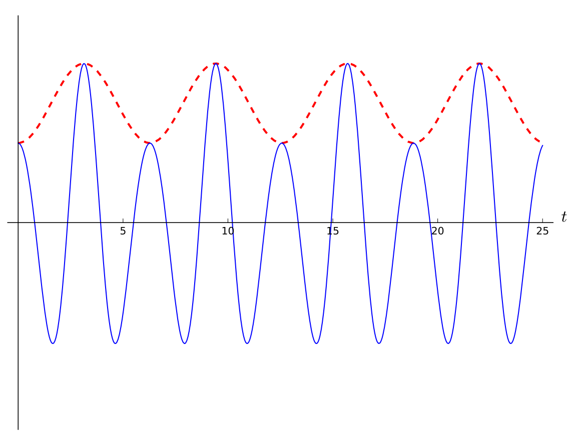
$$y(t) = \left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t) + \left(\beta + \frac{1}{5}\sin(t)\right)\sin(2t).$$

We interpret the function

$$\left(\alpha - \frac{1}{5}\cos(t)\right)\cos(2t)$$

as the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{5}\cos(t)$*. Thus we see that the solutions* $y(t)$ *oscillate at the frequency associated to the homogeneous equation, but that the amplitudes of these oscillations themselves oscillate at the frequency that is the difference between the forcing frequency and the homogeneous frequency.*

A typical solution looks something like the following:



Here the red dashed line is the graph of the function $\alpha - \frac{1}{5}\cos(t)$*, and the blue solid line is the graph of the function* $(\alpha - \frac{1}{5}\cos(t))\cos(2t)$*.*

The previous example illustrates some interesting interplay between the frequency at which the homogeneous solution oscillates and the frequency of the forcing. In order to discuss this interplay, we introduce some terminology. Let ω_h be the frequency at which solutions to the homogeneous equation oscillate; we call ω_h the ***natural*** frequency. Furthermore, let ω_f be the frequency present in the forcing function; we call ω_f the ***forcing frequency***. The previous example shows that we can expect solutions to a forced oscillator with trigonometric forcing to oscillate at frequency ω_h, and that we can expect the amplitude of these oscillations to themselves oscillate with frequency $\omega_f - \omega_h$.

The following two examples illustrate what happens when the forcing frequency is either very far from, or very close to, the natural frequency.

Example 6.5.2. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(30t).$$

In this case, the forcing frequency $\omega_f = 30$ *is very far from the natural frequency* $\omega_h = 2$*.*

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha\cos(2t) + \beta\sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(30t)$*. Plugging this in to the equation we obtain*

$$-90a\cos(30t) + 4a\cos(30t) = \cos(30t).$$

Thus we choose $a = -1/86$ *and obtain the particular solution*

$$y_p(t) = -\frac{1}{86}\cos(30t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{86}\cos(30t).$$

Using the trigonometric identity

$$\cos(30t) = \cos(28t)\cos(2t) - \sin(28t)\sin(2t)$$

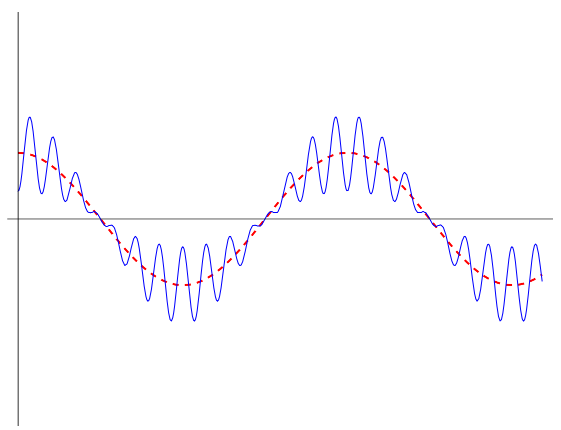
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t) + \left(\beta + \frac{1}{86}\sin(28t)\right)\sin(2t)$$

We can interpret the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

to be the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{86}\cos(28t)$. *Notice that the frequency at which the amplitude is changing is much faster than the natural frequency of the oscillator. Thus a typical solution looks something like the following:*



Here the graph of the function

$$\left(\alpha - \frac{1}{86}\cos(28t)\right)\cos(2t)$$

is shown with a solid blue curve, while the graph of the function $\alpha\cos(2t)$ *is shown with a red dashed curve*

Example 6.5.3. *Consider the forced oscillator equation*

$$\frac{d^2y}{dt^2} + 4y = \cos(2.1t).$$

In this case, the forcing frequency $\omega_f = 2.1$ *is very close to the natural frequency* $\omega_h = 2$.

As in the previous example, the homogeneous solution is

$$y_h(t) = \alpha \cos(2t) + \beta \sin(2t).$$

We look for a particular solution of the form $y_p(t) = a\cos(2.1t)$. *Plugging this in to the equation we obtain*

$$-4.41a\cos(2.1t) + 4a\cos(2.1t) = \cos(2.1t).$$

Thus we choose $a = -1/86$ *and obtain the particular solution*

$$y_p(t) = -\frac{1}{0.41}\cos(2.1t).$$

Consequently, the general solution is

$$y(t) = \alpha\cos(2t) + \beta\sin(2t) - \frac{1}{0.41}\cos(2.1t).$$

Using the trigonometric identity

$$\cos(2.1t) = \cos(0.1t)\cos(2t) - \sin(0.1t)\sin(2t)$$

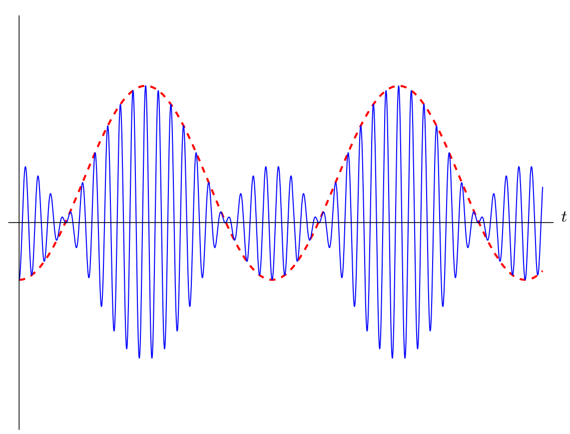
we express the general solution as

$$y(t) = \left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t) + \left(\beta + \frac{1}{0.41}\sin(0.1t)\right)\sin(2t)$$

We can understand the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

as the function $\cos(2t)$ *with amplitude given by* $\alpha - \frac{1}{0.41}\cos(0.1t)$. *Notice that the frequency changes of the amplitude is much smaller than the frequency of the natural oscillations. This gives rise to solutions that look like the following:*



Here the graph of the function

$$\left(\alpha - \frac{1}{0.41}\cos(0.1t)\right)\cos(2t)$$

is shown as a solid blue curve, while the graph of the amplitude function $\alpha - \frac{1}{0.41}\cos(0.1t)$ *is shown as a dashed red curve. The clusters of oscillations shown in this graph are known as* ***beats****, and are typical of cases when the forcing frequency is very close to the natural frequency.*

Beats are frequently used by musicians to tune stringed instruments: Two strings being "in tune" mean that they oscillate at the same natural frequency. If the two strings are very slightly out of tune, then playing one of the strings will have the effect of forcing the other string at a frequency nearby to its natural frequency, resulting in the formation of beats in the vibrations of the second string.

In all of the previous examples, we were able to find a particular solution that was a multiple of the forcing function. Unfortunately, as the following example illustrates, that does not work when there is damping present. However, we are still able to guess a particular solution by considering both cosine and sine functions.

Example 6.5.4. *Consider the equation*

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 4y = \cos(3t),$$

which describes an underdamped oscillator with periodic forcing.

The characteristic equation for the homogeneous equation is

$$\lambda^2 + \lambda + 4 = 0,$$

which has solutions

$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\, i.$$

Thus the homogeneous solution is

$$y_h(t) = \alpha e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}\, t\right) + \beta e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}\, t\right).$$

In order to find a particular solution, we guess that $y_p(t) = a\cos(3t) + b\sin(3t)$*. Plugging this in to the original equation gives us*

$$\{-9a + 3b + 4a\}\cos(3t) + \{-9b - 3a + 4b\}\sin(3t) = \cos(3t).$$

Thus in order to obtain a solution we need

$$-5a + 3b = 1 \qquad \textit{and} \qquad -3a - 5b = 0.$$

Thus we need $a = -5/34$ *and* $b = 3/34$*. The resulting particular solution is*

$$y_p(t) = -\frac{5}{34}\cos(3t) + \frac{3}{34}b\sin(3t).$$

Notice that when guessing our particular solution, we cannot have $b = 0$*. This means that we could not have constructed a particular solution with only* $\cos(3t)$ *in it. (It is a good exercise to try... what goes wrong?)*

Exercise 6.5.1. The equation

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t)$$

models frictionless oscillations with periodic forcing.

1. Find the general solution to the homogeneous equation

$$\frac{d^2y}{dt^2} + 9y = 0.$$

2. Solve the (homogeneous) initial value problem

$$\frac{d^2y}{dt^2} + 9y = 0, \qquad y(0) = 0, \quad y'(0) = 5.$$

3. Find a particular solution of to the inhomogeneous equation

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

4. Find the general solution of the equation

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t).$$

5. Solve the IVP

$$\frac{d^2y}{dt^2} + 9y = 5\sin(2t) - 10\cos(2t), \qquad y(0) = 0, \quad y'(0) = 5.$$

6. Graph the solution of the IVP in the ty-plane, paying particular attention to long-term behavior of the graph.

7. Describe (in words!) the effect the forcing has on the oscillator.

Exercise 6.5.2. Repeat Exercise 6.5.1 for the initial value problem

$$\frac{d^2y}{dt^2} + 16y = 7\sin(3t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Exercise 6.5.3. Repeat Problem 6.5.1 for the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5\sin(t) \qquad y(0) = 0, \quad y'(0) = 0.$$

Exercise 6.5.4. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 16y = \cos 25t, \qquad y(0) = 0, \qquad y'(0) = \frac{1}{100}.$$

Plot the solution and describe its characteristic features.

6.6 Resonance and oscillators

In this section we continue our study of oscillators with trigonometric forcing. Our goal is to understand what happens when the forcing frequency approaches the natural frequency of the oscillator. To do this, we construct an oscillator equation with natural frequency ω and forcing frequency ω_f as follows:

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega_f t). \tag{6.6.1}$$

Our plan is the following: We consider (6.6.1) with initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0 \tag{6.6.2}$$

in the situation that $\omega_f \neq \omega$. Then we take the limit as $\omega_f \to \omega$ and see what happens. (The reason for choosing initial conditions (6.6.2) is that we want to focus attention on the effects of the forcing.)

It is straightforward to see that the homogeneous solution to (6.6.1) is

$$y_h(t) = \alpha\cos(\omega t) + \beta\sin(\omega t).$$

As in the previous section, we now proceed by looking for a particular solution of the form $y_p(t) = a\cos(\omega_f t)$. Plugging this in to (6.6.1), obtain

$$(\omega^2 - \omega_f^2)a\cos(\omega_f t) = \cos(\omega_f t). \tag{6.6.3}$$

Since we are assuming that $\omega_f \neq \omega$, we obtain the particular solution

$$y_p(t) = \frac{1}{\omega^2 - \omega_f^2} \cos(\omega_f t)$$

and thus see that the general solution is

$$y(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) + \frac{1}{\omega^2 - \omega_f^2} \cos(\omega_f t). \tag{6.6.4}$$

We now enforce the initial conditions (6.6.2), computing

$$y'(t) = -\omega\alpha \sin(\omega t) + \omega\beta \cos(\omega t) - \frac{\omega_f}{\omega^2 - \omega_f^2} \sin(\omega_f t).$$

Thus the initial conditions require

$$0 = \alpha + \frac{1}{\omega^2 - \omega_f^2} \qquad \text{and} \qquad 0 = \beta.$$

Consequently, the solution to (6.6.1) – (6.6.2) is

$$y(t) = \frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2}. \tag{6.6.5}$$

We now want to take the limit of (6.6.5) as $\omega_f \to \omega$. Notice that both the numerator and the denominator are zero in the limit. Thus it is appropriate to apply l'Hôpital's rule. Keeping in mind that the variable with which we are taking the limit is ω_f, we find that

$$\lim_{\omega_f \to \omega} \left[\frac{\cos(\omega_f t) - \cos(\omega t)}{\omega^2 - \omega_f^2} \right] = \lim_{\omega_f \to \omega} \left[\frac{-t \sin(\omega_f t)}{-2\omega_f} \right] = \frac{t}{2\omega} \sin(\omega t).$$

Thus in the limit as the forcing frequency approaches the natural frequency, the solution approaches a sine wave that oscillates at the natural frequency and has a linearly growing amplitude.

In order to understand what's going on here, it is useful to recall the concept of *beats* from Example 6.5.3. In that example, we used a trigonometric identity to rewrite the solution in a way that we could interpret as oscillations at the natural frequency with oscillating amplitude. In order to apply that approach here, we use the identity

$$\cos(A) - \cos(B) = 2 \sin\left(\frac{B + A}{2}\right) \sin\left(\frac{B - A}{2}\right)$$

in order to write the solution (6.6.5) as

$$y(t) = \frac{1}{\omega^2 - \omega_f^2} \sin\left(\frac{\omega_f - \omega}{2} t\right) \sin\left(\frac{\omega_f + \omega}{2} t\right).$$

We interpret this last expression as being sinusoidal oscillations of frequency $(\omega_f + \omega)/2$ with periodic amplitude given by

$$\frac{2}{\omega^2 - \omega_f^2} \sin\left(\frac{\omega_f - \omega}{2} t\right).$$

Thus as the forcing frequency approaches the natural frequency, we see that the frequency of the oscillations approaches the natural frequenc; and that the magnitude of the amplitude function increases, while the frequency of the amplitude function decreases. In other words, as ω_f approaches ω, the solution (6.6.5) consists of increasingly large and slow beats of oscillations at near-natural frequency. In the limit, the first beat "takes over" the solution and we an amplitude function that is simply growing linearly.

The following code demonstrates the limit as $\omega_f \to \omega$ nicely:

```
var('t')
@ interact
def f(wf = slider(1.0000001,1.5, default=1.5, label='$\
    omega_f$')):
    solnplot = plot( ( cos(wf*t) - cos(t) )/(1-wf^2), (t
        ,0,120))
    ampplot1 = plot( 2*sin( t*(wf-1)/2)/(1-wf^2)  , (t
        ,0,120), linestyle='dashed', color='red',thickness
        =2)
    ampplot2 = plot( -2*sin( t*(wf-1)/2)/(1-wf^2)  , (t
        ,0,120), linestyle='dashed', color='red',thickness
        =2)
    mainplot = solnplot + ampplot1 + ampplot2
    mainplot.show(ymin=-10,ymax=20)
```

The code can also be accessed via this link.

The picture generated by the code above illustrates the consequences of changing the forcing frequency: As the forcing frequency becomes the closer the natural frequency, the natural response of the system grows dramatically in size, ultimately approaching the linearly growing function

$$y_p(t) = \frac{t}{2\omega} \sin(\omega t). \tag{6.6.6}$$

The situation where the forcing frequency is the same as the natural frequency is an example of ***resonance***. Physically, resonance is when the forcing is "tuned" to the natural frequency of the system. This results in oscillations with amplitudes that grow.

Mathematically, we see resonance occur when there is some sort of degeneracy in the system. In the case of the forced oscillator in which the forcing frequency matches the natural frequency, this degeneracy manifests itself in the fact that we do not find a particular solution of the same form as the forcing. Rather, we find a particular solution that is of the form $t \cdot y_h(t)$. This is explored in greater detail in the next section.

Exercise 6.6.1. In this problem we consider directly the forced oscillator equation (6.6.1) when $\omega_f = \omega$:

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega t). \tag{6.6.7}$$

1. Show that looking for a particular solution to (6.6.7) of the form $y_p(t) = a\cos(\omega t)$ yields an equation that cannot be satisfied.

2. Show by direct computation that (6.6.6) is actually a particular solution to (6.6.7).

3. Find the general solution to (6.6.7).

Exercise 6.6.2. Find the general solution to

$$\frac{d^2y}{dt^2} + 4y = 3\cos(2t).$$

Exercise 6.6.3. In this exercise, we study another type of resonance Consider the equation

$$\frac{d^2y}{dt^2} - 2b\frac{dy}{dt} + y = 0, \tag{6.6.8}$$

where b is some parameter with $0 \leq b \leq 1$.

1. Find the general solution to (6.6.8) when $b = 0$.

2. Now assume that $0 < b < 1$. Find the solution to (6.6.8) satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.

3. Show that in the limit as $b \to 1$ we have $y(t) \to te^t$.

4. Now set $b = 1$ in the equation (6.6.8) and verify that te^t is a particular solution.

5. Find the general solution to (6.6.8) in the case when $b = 1$.

6. In the case that $b = 1$, we say that (6.6.8) is *resonant*. Why is the term "resonant" appropriate in this case?

Exercise 6.6.4. Find the general solution to

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0.$$

6.7 General resonance

As alluded to in the previous section, the mathematical phenomenon of resonance corresponds to a degeneracy in the system. In this section we explore two different types of degeneracies: repeated eigenvalues and degenerate forcing. In both cases, the degeneracy appears when the "usual" methods yield fewer solutions than we were expecting to find. In both cases, we can find the "remaining" solutions in the form of

$$t \cdot (\text{those solutions we were able to find}).$$

I refer to this method as the "t-trick."

The first type of resonance we study is the case of repeated eigenvalues. An example of this was explored in Exercise 6.6.3. More generally, suppose we have a differential equation of the form

$$\frac{d^2y}{dt^2} - 2\mu\frac{dy}{dt} + \mu^2 y = 0. \tag{6.7.1}$$

The characteristic equation for this ODE is

$$\lambda^2 - 2\mu\lambda + \mu^2 = 0,$$

which we write as

$$(\lambda - \mu)^2 = 0.$$

Thus there is only one eigenvalue, namely $\lambda = \mu$. From this we know that $y_1(t) = e^{\mu t}$ is one solution to (6.7.1). However, in order to use the superposition principle to find the general solution we need to find a second solution. Motivated by Exercise 6.6.3, we investigate whether $te^{\mu t}$ is a solution. We compute

$$\frac{d^2}{dt^2}\left[te^{\mu t}\right] - 2\mu\frac{d}{dt}\left[te^{\mu t}\right] + \mu^2\left(te^{\mu t}\right) = 0$$

and thus conclude that $y_2(t) = te^{\mu t}$ is indeed a solution, and therefore that the general solution to (6.7.1) is

$$y(t) = \alpha e^{\mu t} + \beta te^{\mu t}.$$

Example 6.7.1. *Suppose we want to solve the initial value problem*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 3.$$

If we look for solutions of the form $e^{\lambda t}$ we find that λ must satisfy the characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0.$$

The only such λ is $\lambda = 4$ and thus we find only the solution e^{4t}. However, the function te^{4t} is also a solution, and thus by superposition the general solution is

$$y(t) = \alpha e^{4t} + \beta te^{4t}.$$

In preparation for imposing the initial conditions, we compute

$$y'(t) = 4\alpha e^{4t} + \beta e^{4t} + 4\beta te^{4t}.$$

Thus the initial conditions reduce to

$$1 = \alpha \qquad \text{and} \qquad 3 = 4\alpha + \beta.$$

Consequently, we find that the solution to the initial value problem is

$$y(t) = e^{4t} - te^{4t}.$$

Activity 6.7.1. *Find the solution to the initial value problem*

$$9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = 7, \qquad y(0) = 1, \qquad y'(0) = 5.$$

The second type of resonance we consider is forced equations for which our "usual guess" for the particular solution does not work. This happens when, for example, the forcing term is of the same term as the homogeneous solution. When this occurs, we make the "educated guess" that the particular solution takes the form $t \cdot y_h(t)$.

Example 6.7.2. *Consider the differential equation*

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}.$$

We first address the homogeneous equation

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

by solving the characteristic equation

$$\lambda^2 + 7\lambda + 10 = 0.$$

We find that $\lambda = -2$ and $\lambda = -5$ are solutions, from which we deduce that the homogeneous solution is

$$y_h(t) = \alpha e^{-2t} + \beta e^{-5t}.$$

Notice that the forcing function $f(t) = e^{-2t}$ is of the same form as the homogeneous solution. In particular, if we go looking for a particular solution of the form $y_p(t) = ae^{-2t}$ we obtain the equation

$$0 = e^{-2t},$$

which is a contradiction.

Instead, we use the t-trick, and go looking for a particular solution of the form

$$y_p(t) = ate^{-2t}.$$

Plugging this in to the original equation yields

$$\frac{d^2}{dt^2}\left[ate^{-2t}\right] + 7\frac{d}{dt}\left[ate^{-2t}\right] + 10\left(ate^{-2t}\right) = e^{-2t},$$

which simplifies to

$$3ae^{-2t} = e^{-2t}.$$

Thus we choose $a = 1/3$ and have particular solution

$$y_p(t) = \frac{1}{3}te^{-2t}.$$

Using this, we find the general solution

$$y(t) = \alpha e^{-2t} + \beta e^{-5t} + \frac{1}{3}te^{-2t}.$$

Sometimes we get to use the t-trick twice, as the following activity illustrates!

Activity 6.7.2. *Find the general solution to the equation*

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}.$$

Exercise 6.7.1. Find the general solution of the following ODEs:

1. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 0;$
2. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 1 + t + e^{2t};$
3. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 9y = 0;$
4. $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 4y = e^{4t};$
5. $\dfrac{d^2y}{dt^2} + 9y = \sin(2t);$
6. $\dfrac{d^2y}{dt^2} + 9y = 3\cos(3t).$

Writing assignment 4: Damped oscillators.

In this writing assignment you analyze the damped oscillator equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0. \tag{$\heartsuit$}$$

The main goal of your report is to explain how the value of the damping coefficient b impacts the behavior of solutions.

- Begin your report with a description of solutions to $(\heartsuit)$ in the case that $b = 0$. This is called the ***undamped*** setting.

  In this part of the report you should indicate how the values of k and m affect solutions. Do this by making statements of the form *When m is large relative to k, the solutions...*

  Note: To type $m \ll k$ or $m \gg k$, use the code `$m\ll k$` or `m\gg k$`.

- For the remainder of the report, assume $b > 0$. You should find that for certain values of b, solutions to $(\heartsuit)$ do not oscillate at all. This is called the ***overdamped*** case. The other cases are called ***underdamped*** and ***critically damped***. Describe all three cases in as much detail as you can.

- Your report should include graphics that illustrate the behavior of typical solutions. In order to generate the graphics, you need to choose some actual numbers. However, your discussion should be entirely in terms of m, b, and k. Thus the reader will never know the numbers that you used to generate the plots.