Chapter 5

Nonlinear first order systems

The goal of this chapter is to rigorously demonstrate the qualitative behavior of solutions to nonlinear systems of the form

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y). \tag{5.0.1}$$

Our analysis has two aspects:

- We understand global features of the phase portrait associated to (5.0.1) using *nullclines*. These are curves that divide the phase plane in to regions for which we understand precisely where solution curves enter and exit.

- We understand the stability of equilibrium points by linearizing (5.0.1) about each, and subsequently using the theory from the previous chapter.

Before we begin, it is useful to have some simple examples in mind.

Example 5.0.1. *The following is a simple predatory-prey model:*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \qquad \frac{dy}{dt} = -y + xy.$$

Example 5.0.2. *The following is a simple species competition model:*

$$\frac{dx}{dt} = 8x\left(1 - \frac{x}{8}\right) - xy \qquad \frac{dy}{dt} = 4y - 2xy.$$

5.1 Nullclines

In this section we develop a tool for understanding the large-scale behavior of nonlinear systems. Suppose we have a system of the form (5.0.1). The vector field

$$\langle f(x,y), g(x,y)\rangle$$

describes the motion of solutions in the x–y phase plane.

A ***nullcline*** is a curve in the phase plane where the vector field defined by the differential equation points in a particular direction. For systems of the form (5.0.1), we focus on two special cases:

Vertical motion nullclines are locations in the phase plane where $\frac{dx}{dt} = 0$. This corresponds to points (x, y) such that $f(x, y) = 0$.

Horizontal motion nullclines are locations in the phase plane where $\frac{dy}{dt} = 0$. This corresponds to points (x, y) such that $g(x, y) = 0$.

The vertical and horizontal motion nullclines divide the phase plane in to regions. Along the boundary of these regions we know that solutions are either moving horizontally or vertically, depending on which type of nullcline that boundary is. Using the differential equation, we can also determine which direction solutions are moving along various sections of a nullcline. The result is that we understand, at a broad level, the path that solutions may take from one region to the next.

Example 5.1.1. *Consider the system from Example 5.0.1:*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \qquad \frac{dy}{dt} = -y + xy.$$

The vertical motion nullclines are given by the condition

$$0 = 2x\left(1 - \frac{x}{2}\right) - xy = x\left(2 - x - y\right).$$

Thus we have two vertical motion nullclines:

$$x = 0 \qquad \textit{and} \qquad 2 - x - y = 0.$$

We first focus on the nullcline $x = 0$, which corresponds to the y axis. Along this nullclines, the vertical motion is described by

$$\frac{dy}{dt} = -y + (0)(y) = -y.$$

Thus we see that the motion of solutions along the y axis is upward when $y < 0$ and downward when $y > 0$. There is no motion when $y = 0$.

From this we deduce the following:

- *Since the motion along the nullcline $x = 0$ is vertical, and the nullcline itself is a vertical line, no solutions can cross this nullcline.*
- *The point $(x, y) = (0, 0)$ must be an equilibrium point, since there is no motion in either x or y directions.*

We now focus on the nullcline $2-x-y = 0$, which is a straight line in the phase plane. Along this this nullcline, the vertical motion can be deduced by writing the line as $y = 2 - x$ and substituting in to the differential equation for y. Thus the vertical motion is given by

$$\frac{dy}{dt} = -(2 - x) + x(2 - x) = -(x - 2)(x - 1),$$

which is positive when $1 < x < 2$ and is otherwise negative. Thus we see that solutions curves in the phase plane crossing the line $y = 2 - x$ must cross it in the upward direction when $1 < x < 2$ and otherwise must cross it in the downward direction. Notice that there is no motion at points where $x = 1$ or $x = 0$. Thus we must have equilibrium points at $(x, y) = (1, 1)$ and $(x, y) = (2, 0)$.

We now turn to the horizontal motion nullclines, which occur when

$$0 = -y + xy = y(x - 1).$$

Thus we have two horizontal motion nullclines, at

$$y = 0 \qquad \text{and} \qquad x = 1.$$

The nullcline $y = 0$ is the x axis. The horizontal motion along this nullcline is given by

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) = x(2 - x).$$

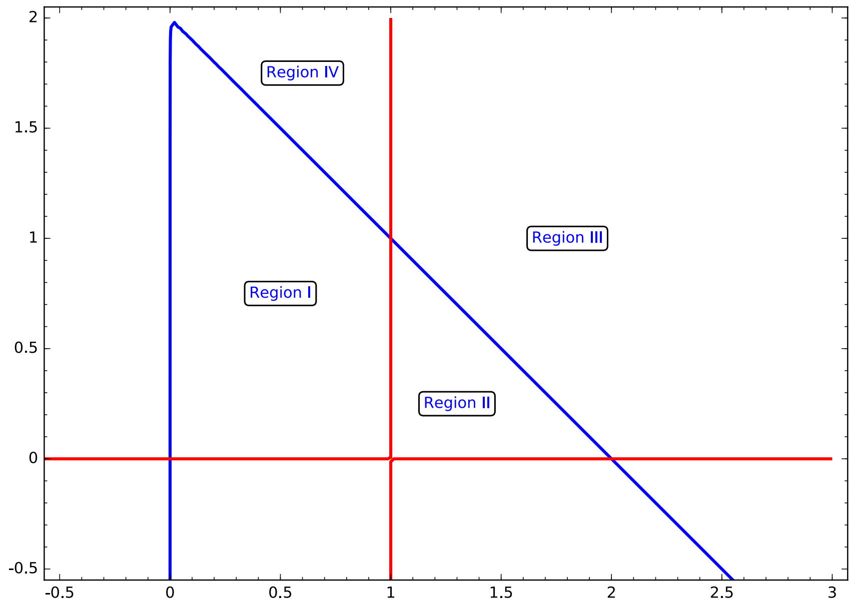
Thus we see that motion is to the right when $0 < x < 2$ and is otherwise to the left. Since the motion along $y = 0$ is horizontal, no solutions can cross this nullcline.

The nullcline $x = 1$ is a vertical line passing through the equilibrium at $(1, 1)$. The horizontal motion along this nullcline is given by

$$\frac{dx}{dt} = 2(1)\left(1 - \frac{1}{2}\right) - (1)y = 1 - y.$$

Thus when $y < 1$ the horizontal motion of solutions in the phase plane along this nullcline is to the right, while when $y > 1$ the horizontal motion of solutions is to the left.

In combination, the four nullclines above divide the phase plane in to a number of regions. Since the system is a population model, we focus on those regions in the first quadrant, labeling them as follows:



We now use this diagram to understand, in a very rough way, the trajectory of typical solutions. First consider Region I. Solutions cannot cross the nullclines $x = 0$ or $y = 0$, and the motion across the boundary between Regions I and IV is downward. Thus a solution in Region I either stays there (perhaps tending towards one of the equilibria) or pass through the horizontal motion nullcline in to Region II.

Solutions in Region II can either tend to one of the equilibrium points, or pass through the vertical motion nullcline in to Region III.

At first sight, we might tend towards one of the equilibrium points or might tend off to infinity, but if they leave they must do so by crossing the horizontal motion nullcline in to Region IV. In fact, for a solution in Region III we have $y > 2 - x$ and $x > 0$. Thus for such a solution we have $-xy < -x(2-x)$ and hence

$$\frac{dx}{dt} < 2x\left(1 - \frac{x}{2}\right) - x(2-x) = 0.$$

Thus solutions in Region III are always moving to the left, making it more likely that they will cross in to Region IV.

Solutions in Region IV satisfy $0 < x < 1$ *and* $y > 0$. *Thus their vertical motion satisfies*

$$\frac{dy}{dt} = -y + xy < -y + y = 0.$$

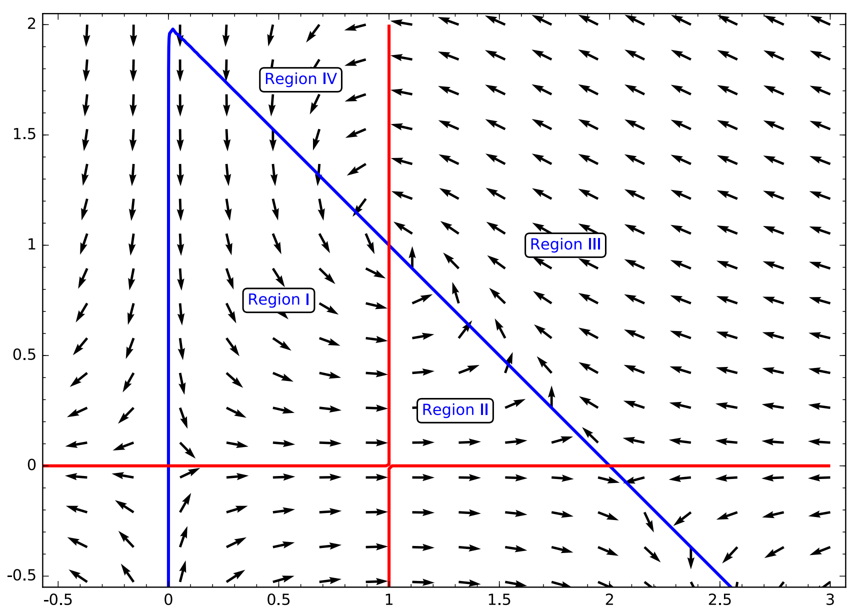
Thus solutions in this region are always moving downward. If they leave Region IV, it must be by crossing the vertical motion nullcline in to Region I.

In summary, it seems that solutions in the first quadrant must do one of the following:

- *tend towards an equilibrium solution,*
- *tend to infinity in the vertical direction in Region III, or*
- *follow a cycle*

  Region I $\to$ *Region II* $\to$ *Region III* $\to$ *Region IV* $\to$ *Region I* $\to$ *etc.*

In fact, if we superimpose the vector field plot on top of the nullclines, we see that this cyclical behavior is precisely what we expect from solutions!

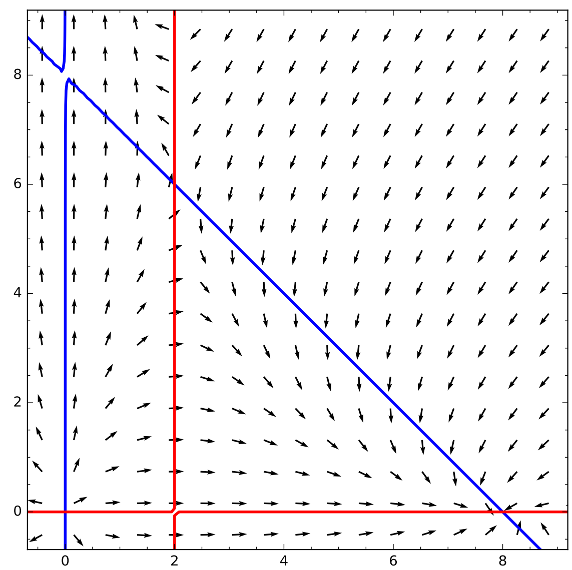


Using nullclines, we can get a good sense of how solutions behave away from equilibrium points. However, the pictures in the previous example don't give us a very good sense of what happens near equilibrium points. Thus we turn to the analysis of equilibrium points next.

Activity 5.1.1. *Construct the nullclines for the system*

$$\frac{dx}{dt} = 8x\left(1 - \frac{x}{8}\right) - xy \qquad \frac{dy}{dt} = 4y - 2xy$$

appearing in Example 5.0.2. You should get a picture that looks roughly like this:



Exercise 5.1.1. Extract as much information as possible about the system

$$\frac{dx}{dt} = x - xy \qquad \frac{dy}{dt} = -2y + 2xy$$

in Exercise 5.2.1 by using the method of nullclines.

Exercise 5.1.2. The following system of equations models the populations (in millions) of two competing animal species:

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \\ \\ \dfrac{dy}{dt} = 4y\left(1 - \frac{y}{4}\right) - 3xy. \end{cases}$$

Extract as much information as possible about the system using the method of nullclines.

5.2 Linearization & Equilibrium point analysis

We continue our task of understanding the behavior of solutions to equations of the form

$$\frac{dx}{dt} = f(x,y) \qquad \text{and} \qquad \frac{dy}{dt} = g(x,y). \tag{5.2.1}$$

In the previous section we saw how to use nullclines in order to understand behavior on the large scale. In this section we explore how to understand the behavior of solutions in a neighborhood of an equilibrium point.

Suppose that (x_*, y_*) is an equilibrium point for the system (5.2.1), meaning that the constant functions $x(t) = x_*$ and $y(t) = y_*$ are solutions. Then the constants x_*, y_* satisfy

$$f(x_*, y_*) = 0 \qquad \text{and} \qquad g(x_*, y_*) = 0. \tag{5.2.2}$$

Suppose now that (x, y) is close to (x_*, y_*). Then the Taylor approximation theorem says that

$$\begin{aligned} f(x,y) &\approx f(x_*,y_*) + \frac{\partial f}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial f}{\partial y}(x_*,y_*)\,(y-y_*) \\ &\frac{\partial f}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial f}{\partial y}(x_*,y_*)\,(y-y_*), \\ g(x,y) &\approx g(x_*,y_*) + \frac{\partial g}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial g}{\partial y}(x_*,y_*)\,(y-y_*) \\ &\frac{\partial g}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial g}{\partial y}(x_*,y_*)\,(y-y_*), \end{aligned}$$

where we have used (5.2.2). Thus any solution to (5.2.1) that is close to the equilibrium point (x_*, y_*) for some period of time, then we have

$$\begin{aligned} \frac{d}{dt}\left[x - x_*\right] &\approx \frac{\partial f}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial f}{\partial y}(x_*,y_*)\,(y-y_*), \\ \frac{d}{dt}\left[y - y_*\right] &\approx \frac{\partial g}{\partial x}(x_*,y_*)\,(x-x_*) + \frac{\partial g}{\partial y}(x_*,y_*)\,(y-y_*), \end{aligned} \tag{5.2.3}$$

with the approximation valid as long as the solution (x, y) is close to (x_*, y_*).

We now discuss how to use the approximation (5.2.3) to study the stability of the equilibrium point at (x_*, y_*). We first introduce the ***linearization of*** (5.2.1) ***at*** $\boldsymbol{(x_*, y_*)}$

$$\begin{aligned} \frac{dv}{dt} &= \frac{\partial f}{\partial x}(x_*,y_*)\,v + \frac{\partial f}{\partial y}(x_*,y_*)\,w, \\ \frac{dw}{dt} &= \frac{\partial g}{\partial x}(x_*,y_*)\,v + \frac{\partial g}{\partial y}(x_*,y_*)\,w. \end{aligned} \tag{5.2.4}$$

Notice that (5.2.4) is a linear equation of the type studied in the previous chapter; we can write it in matrix form as

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_*, y_*) & \frac{\partial f}{\partial y}(x_*, y_*) \\ \\ \frac{\partial g}{\partial x}(x_*, y_*) & \frac{\partial g}{\partial y}(x_*, y_*) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Suppose now that (x, y) is a solution to (5.2.1) that is close to (x_*, y_*) for some time interval. Then during that time interval we have that

$$x \approx x_* + v \qquad \text{and} \qquad y \approx y_* + w$$

for some (v, w) satisfy (5.2.4).

The upshot of all this is that we can view solutions (v, w) to (5.2.4) as telling us, approximately, how the displacement from equilibrium evolves for solutions to (5.2.1). Although this linearization is an approximation, it is an increasingly better approximation as we get closer to the equilibrium point. The Taylor remainder theorem tells us that when we are close enough to the equilibrium point, the linear part of the Taylor approximation dominates (unless the linear part is zero), and thus we can indeed the stability of the equilibrium point (x_*, y_*) with regards to the system (5.2.1) is the same as the stability of $(0, 0)$ with regards to the system (5.2.4).

To see how this works in practice, consider the following example.

Example 5.2.1. *Consider the simple predator-prey model from Example 5.0.1:*

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \qquad \frac{dy}{dt} = -y + xy.$$

It is easy to see that the equilibrium points are

$$(0, 0), \qquad (2, 0), \qquad (1, 1).$$

In order to linearize the equation about each of these equilibrium points we set

$$f(x, y) = 2x\left(1 - \frac{x}{2}\right) - xy = 2x - x^2 - xy$$
$$g(x, y) = -y + xy$$

and compute

$$\frac{\partial f}{\partial x} = -2x - y + 2$$
$$\frac{\partial f}{\partial y} = -x$$
$$\frac{\partial g}{\partial x} = y$$
$$\frac{\partial g}{\partial y} = x - 1$$

The linearization of the system at $(x, y) = (1, 1)$ *is*

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}.$$

We compute the eigenvalues of this matrix to be

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

The equilibrium $(v, w) = (0, 0)$ *is a counter-clockwise spiral sink, and is stable. From this we conclude that the equilibrium* $(x, y) = (1, 1)$ *is a stable counter-clockwise spiral sink.*

The linearization of the system at $(x, y) = (0, 0)$ *is*

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}.$$

We compute the eigenvalues for this matrix to be

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = 2.$$

The equilibrium $(v, w) = (0, 0)$*, and thus the equilibrium* $(x, y) = (0, 0)$*, is an unstable saddle. We can furthermore compute the eigenvectors, finding that the general solution to the linearization is*

$$\begin{pmatrix} v \\ w \end{pmatrix} = \alpha e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus solutions (x, y) *nearby to* $(0, 0)$ *will be attracted to the equilibrium in the* y *direction, but repelled in the* x *direction.*

Finally, we linearize the system at the equilibrium $(x, y) = (2, 0)$*. The linearization is*

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}.$$

We compute the eigenvalues to be

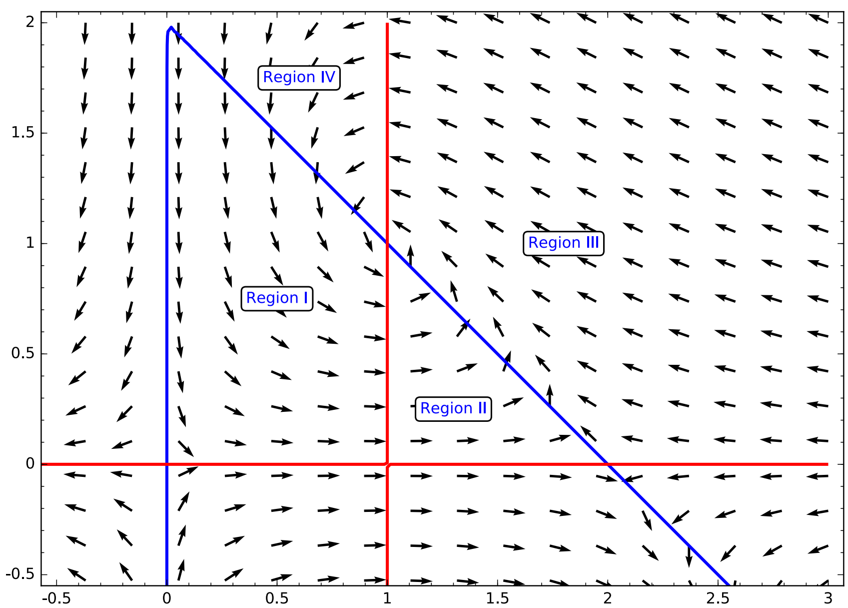
$$\lambda_1 = -2 \qquad \text{and} \qquad \lambda_2 = 1.$$

Thus this equilibrium is an unstable saddle. We can compute the general solution to be

$$\begin{pmatrix} v \\ w \end{pmatrix} = \alpha e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Thus we see that solutions (x, y) approach the equilibrium $(2, 0)$ along the x axis, and then are pushed away in the direction $\pm\langle 2, -3\rangle$.

We can perhaps best understand all this in the context of the nullcline picture we obtained in the previous section:



Our equilibrium point analysis shows that solutions in Region I that are close to the equilibrium point $(x, y) = (1, 1)$ will indeed spiral in to that equilibrium point. Solutions in Region I that are close to the equilibrium at $(0, 0)$ are pushed away from that equilibrium. We also see that solutions in Region II that approach $(x, y) = (2, 0)$ will be pushed in to Region III in the direction $\langle -2, 3\rangle$.

Activity 5.2.1. *Consider the competition model*

$$\frac{dx}{dt} = 8x\left(1 - \frac{x}{8}\right) - xy \qquad \frac{dy}{dt} = 4y - 2xy$$

from Example 5.0.2. Linearize this equation about each equilibrium point. Use the linearizations to both understand the stability of each equilibrium point as well as make sense of what happens at the "corners" of each nullcline region.

Exercise 5.2.1. Perform the equilibrium point analysis of the following predator-prey model:

$$\frac{dx}{dt} = x - xy \qquad \frac{dy}{dt} = -2y + 2xy$$

That is, execute the following steps.

- Find all equilibrium solutions of the system.
- Linearize the system near each equilibrium.
- Understand the linearized models using eigenstuff.
- As much as possible, piece the phase portraits of the linearized systems together to get an approximate phase portrait of the full system.
- Find the nullclines of the system. Use a combination of the nullclines and the equilibrium point analysis in order to obtain as complete an understanding of how solutions behave as you can.
- Use `Sage` to construct a plot of the vector field for the system. Draw the nullclines and equilibrium points on your plot using a pencil.
- Write a couple sentences describing how "typical" solutions behave.

Exercise 5.2.2. Repeat Exercise 5.2.1 for the predator-prey model

$$\frac{dP}{dt} = 0.3P\big(1 - \frac{P}{100}\big) - 0.06PR \qquad \frac{dR}{dt} = -0.4R + 0.01PR.$$

Exercise 5.2.3. Repeat Exercise 5.2.1 for the predator-prey model:

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{100}\right) - 0.005xy \qquad \frac{dy}{dt} = \frac{y}{2}\left(1 - \frac{y}{200}\right) + 0.01xy.$$

Exercise 5.2.4. Consider the non-linear system

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -x + (1 - x^2 - y^2)y.$$

1. There is one equilibrium solution of this system – find it!
2. Linearize the system near this equilibrium, and draw the phase portrait of the linearized system.
3. Make an educated guess about the phase portrait of the non-linear system. *For your own benefit do this without any help of "technology".*
4. Show that $x(t) = \sin(t)$, $y(t) = \cos(t)$ is a solution of the non-linear system. What is the phase diagram of this solution?
5. Now make an educated guess about the phase portrait of the non-linear system. (Remember: phase curves for nice systems do not intersect!!!)
6. Construct a phrase plot using `Sage`.
7. Comment on what you learned about linearization from this problem.