Chapter 1

Introduction to differential equations

1.1 What is this course about?

A ***differential equation*** is an equation where the unknown quantity is a function, and where the equation involves the derivative(s) of that unknown function. This definition is perhaps a bit abstract, and it's not immediately clear that such objects are useful or interesting. So let's illustrate with an example.

Suppose we want to mathematically describe a population that changes in time. For example, we might want to describe the population of a certain bacteria culture in a jar of yogurt. We can do this with a function $P(t)$ that represents the size of the population at each time t. For the purposes of this example, let's suppose that t is measured in minutes and $P(t)$ is measured in millions of bacteria. The derivative $P'(t)$ tells us the rate of change of the population. It is important to note that the units of $P'(t)$ are the units of P divided by the units of t. A good way to see this is to note that

$$P'(t) = \frac{dP}{dt} = \frac{\text{infinitesimal change in } P}{\text{infinitesimal change in } t}.$$

Thus in our example, $P'(t)$ has units of millions per minute. We might suppose that the rate of change of the population is a constant 7 million per minute. Mathematically, this is described by the relation

$$\frac{dP}{dt} = 7. \tag{1.1.1}$$

The equation (1.1.1) is a differential equation. The unknown object is P, the function describing the population, and the equation involves the first derivative of P. (Note that we have suppressed the units when writing down the differential equation.)

In fact, (1.1.1) is an ***ordinary differential equation (ODE)***, which means that the unknown function P depends only on one variable, namely t. ODEs involve only the "ordinary" derivatives studied in the single-variable calculus course. Differential equations where the unknown function depends on multiple variables involve partial derivatives, and are called ***partial differential equations***. In this course we focus primarily on ordinary differential equations.

We make two remarks about (1.1.1).

- First, the equation (1.1.1) is a statement about functions, not numbers. It states that the function $P'(t)$ is the same function as the constant function whose output is always 7. This means that (1.1.1) is to be interpreted as holding at all times t being considered.
- Second, we highlight the process by which we obtained (1.1.1). We began by describing a physical quantity (the population size) using the function $P(t)$. We then proceeded to make an assumption about the physical quantity (that its rate of change was constant). Subsequently, we translated that assumption in to an equation. This is a standard process for constructing differential equations that describe situations "out in the world."

This course is an introduction to the study of differential equations. The course focuses on the following questions:

Construction of ODEs How can we construct ODEs that describe specific physical situations?

Existence and uniqueness of solutions How do we know whether there are any solutions to a particular ODE? If there are solutions, how many are there?

Qualitative behavior of solutions What do "typical" solutions to an ODE "look like"?

1.2 Constructing differential equations

We constructed the differential equation (1.1.1) by assuming that the derivative of the function $P(t)$ was constant. The derivative $P'(t)$ represents the

absolute growth rate, also called the ***absolute rate of change***, of the function P. Rates of change, however, are often discussed in terms of percentages rather than in absolute terms. Such percentages refer to the ***relative growth rate***, also called the ***percent rate of change***, which is defined by

$$\text{relative growth rate of } P = \frac{P'(t)}{P(t)} = \frac{1}{P}\frac{dP}{dt}.$$

Notice that the units of the relative growth rate are simply 1 divided by the units of t; the units of P cancel out.

We can construct differential equations by making assumptions about either the absolute or relative growth rate of a population. The simplest assumption, which is the one made in the previous section, is that the absolute growth rate is constant. The corresponding differential equation is

$$\frac{dP}{dt} = r, \tag{1.2.1}$$

where r is some constant. The differential equation (1.2.1) is called the ***constant growth model*** because it describes a population with a constant absolute growth rate.

Alternatively, we may assume that the relative growth rate is constant. The resulting differential equation is

$$\frac{1}{P}\frac{dP}{dt} = r, \tag{1.2.2}$$

where r is some constant. The differential equation (1.2.2) is called the ***basic growth model***. The basic growth model describes situations where population P is growing at r percent per unit of time.

Example 1.2.1. *Suppose a fixed amount of money is placed in to a bank account that earns* 3% *annual interest. Let* $A(t)$ *be the value of the account at time* t*, where we measure* t *in years. Then* A *must satisfy*

$$\frac{1}{A}\frac{dA}{dt} = 0.03.$$

Notice that the amount of money originally placed in to the account does not affect this differential equation. The equation only describes how the amount is changing, not what the amount actually is!

Notice that the constant r in (1.2.1) has different units from the constant r appearing in (1.2.2)!

Multiplying both sides of (1.2.2) by P, we can re-write the basic growth model as

$$\frac{dP}{dt} = rP. \tag{1.2.3}$$

We can interpret the equation (1.2.3) as the statement that the absolute rate of change of P is proportional to P. (Remember that "is proportional to" means "is equal to a constant times.") Thus there are two different assumptions that both lead to basic growth model equation.

It is useful, and interesting, to consider variations on the basic growth model that incorporate additional features.

Example 1.2.2. *Suppose that the bacteria population in my jar of yogurt has a relative growth rate of* 5% *per hour. Additionally, I constantly remove yogurt from the jar in such a way that bacteria are being removed at a rate of* 7 *million per hour.*

We can construct differential equation modeling this situation as follows. Let $P(t)$ *be the population of bacteria in the jar at time* t*, measured in millions; let the time* t *be measured in hours. We assume that the absolute rate of change of* P *is the sum of two terms, the first coming from the relative growth rate and the second coming from the removal of the bacteria. The resulting differential equation is*

$$\frac{dP}{dt} = 0.05P - 7. \tag{1.2.4}$$

The term $0.05P$ *comes from the assumption on the relative growth rate of the bacteria, while the term* -7 *comes from the removal of the bacteria. Notice that each of the two terms on the right side of* (1.2.4) *have units of millions per hour, which is the same as the units of the left side. It is always important that terms being added together have the same units!*

More generally, if population modeled by P grows with relative growth rate r and is subject to "migration" determined by the function f, then the P satisfies

$$\frac{dP}{dt} = rP + f. \tag{1.2.5}$$

The function f is sometimes called the ***forcing term***, and thus we can call (1.2.5) the ***forced basic growth model***.

Another important variation on the basic growth model comes from making the assumption that the population being studied lives in a habitat that can only sustain a finite size population, and that the relative growth rate of the population is proportional to the percent of available habitat. In order to construct a differential equation from this assumption, let $P(t)$ describe the size of the population and let K be the population size that the habitat can sustain. The percent of habitat that is available is given by the function

$$1 - \frac{P}{K}.$$

Thus the assumption that the relative growth rate is proportional to the percent of available habitat leads to the differential equation

$$\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right),$$

where r is some constant. We can rewrite this equation as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{1.2.6}$$

The equation (1.2.6) is called the ***logistic growth model***.

Notice that the constant r in the logistic growth model (1.2.6) has the same units as the constant r in the basic growth model (1.2.2). In fact, if P is very small relative to K, then the term P/K in (1.2.6) is very close to zero. Thus when the habitat can support a very large population size (relative to the current population size), then the logistic growth model is approximated by the basic growth model. (We make this idea of one differential equation approximating another a bit more precise in the next chapter.) For this reason we say that the constant r in the logistic model is the ***ideal relative growth rate***, or ***relative growth rate under ideal conditions***.

Example 1.2.3. *Suppose that the milk in my yogurt jar can sustain a population of* 9 *million bacteria, and that the bacteria has ideal relative growth rate of* 15% *per hour. Then we can model the population of bacteria in the jar by the differential equation*

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9}\right),$$

where we measure P in millions and t in hours.

Notice that in the previous example I chose to measure the population P in millions. We could have decided to measure the population in units of individual bacteria, in which case the differential equation would be

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{9000000}\right).$$

Both of these differential equations describe the same physical situation, but the first one is much easier to work with since the numbers are of more "reasonable size." In general, it is advisable to choose units so that the numbers appearing in the resulting differential equations are as "reasonable" as possible.

It is possible to modify the logistic model in ways analogous to the modification of the basic growth model appearing in Example 1.2.2.

Activity 1.2.1. *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 7% *per year and lives in a habitat that can sustain a population of* 12,000 *individuals. Furthermore, individuals are moving in to the habitat at a constant rate of* 3,000 *per year.*

Activity 1.2.2. *Construct a differential equation describing the following situation: A population has ideal relative growth rate of* 3% *per month and lives in a habitat that can sustain a population of* 500 *individuals. Furthermore, individuals are continuously leaving the habitat at a rate of* 5% *per month.*

All of the differential equations constructed above are population models. Differential equations can be used to describe a wide variety of phenomena beyond population dynamics. Later in the course (especially in Chapters 5 and 6) we address a number of equations arising in physics, while in the first part of the course (Chapters 2, 3, and 4) we focus primarily on population models. The concept of a "population," however, can encompass a wide range of phenomena, as the following examples demonstrate.

Example 1.2.4. *Consider a tank with capacity* 100 *gallons, containing a mixture of fresh water and salt. The contents of the tank are being drained at a rate of* 5 *gallons per minute. Simultaneously, salt water having concentration of* 8 *ounces per gallon is being pumped in to the tank. We assume that the mixture in the tank is being stirred so that the salt concentration in the tank is uniform.*

We can model this situation by considering the amount of salt in the tank to be a "population." Let $S(t)$ *be the amount of salt (measured in ounces) in the tank at time* t *(measured in minutes). Then the rate of change of salt in the tank satisfies*

$$\frac{dS}{dt} = (\textit{rate of salt in}) - (\textit{rate of salt out}).$$

We can easily find expressions for the rate of salt entering and leaving the tank as follows:

$$\begin{aligned}(\textit{rate of salt in}) &= (\textit{rate of liquid entering tank}) \\ &\quad \times (\textit{concentration of salt in the incoming liquid}) \\ &= \left(\frac{5 \textit{ gallons}}{\textit{minute}}\right)\left(\frac{8 \textit{ ounces}}{\textit{gallon}}\right) \\ &= 40 \,\frac{\textit{ounces}}{\textit{minute}}\end{aligned}$$

Similarly, we compute that

$$\begin{aligned}(\textit{rate of salt out}) &= (\textit{rate of liquid leaving tank}) \\ &\quad \times (\textit{concentration of salt in the tank}) \\ &= \left(\frac{5 \textit{ gallons}}{\textit{minute}}\right)\left(\frac{S \textit{ ounces}}{100 \textit{ gallons}}\right) \\ &= \frac{1}{20} S \,\frac{\textit{ounces}}{\textit{minute}}\end{aligned}$$

Thus the differential equation that describes the change in S is

$$\frac{dS}{dt} = 40 - \frac{1}{20}S. \tag{1.2.7}$$

Notice that (1.2.7) *takes the form of the forced basic growth model* (1.2.5) *with relative growth rate $r = -1/20$ and forcing $f = 40$.*

Activity 1.2.3. *Paul's brother purchases* 22 *ounce chocolate drink and begins drinking at a rate of* 3 *ounces per minute. Paul begins to pour strawberry drink, which contains* 10% *fruit, in to his brother's glass at a rate of* 1 *ounce per minute. Find a differential equation that models the rate of change of fruit in the brother's glass as a function of time.*

Exercise 1.2.1. Construct a differential equation which models the following situations.

1. An investment $y(t)$ grows with relative growth rate of 5% per year;

2. \$1000 is invested with annual interest rate of 5%;

3. Continuous deposits are made into an account at the rate of \$1000 a year. In addition to these deposits, the account earns 7% interest per year.

4. Paul takes out a loan with an annual interest rate of 6%. Continuous repayments are made totaling \$1000 per year.
5. A fish population under ideal conditions grows at a relative growth rate k per year. The carrying capacity of their habitat is N and H fish are harvested each month.
6. A fish population under ideal conditions grows at growth rate k per year. The carrying capacity of their habitat is N and one quarter of the fish population is harvested annually.
7. Due to the pollution problems the relative growth rate of a fish population is a decreasing exponential function of time.

Exercise 1.2.2. Some quantity f changes with time, is measured in gallons, and is modeled by the differential equation

$$\frac{df}{dt} = k\, f\left(1 - \frac{f}{M}\right).$$

The time variable t is measured in months and the parameter M is also measured in gallons.

1. In what units is $\frac{df}{dt}$ measured?
2. In what units is $1 - \frac{f}{M}$ measured?
3. In what units is $f\left(1 - \frac{f}{M}\right)$ measured?
4. In what units is the parameter k measured?

Exercise 1.2.3. A variable quantity $r = r(t)$ is measured in grams per liter and is modeled by the differential equation

$$\frac{dr}{dt} = -k \cdot \frac{r}{c + r}.$$

The time variable t is measured in seconds. Figure out the units for the parameters k and c.

Exercise 1.2.4. We consider a large urn of coffee in cafeteria. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.

Suppose the following:

- There is initially 4 liters of coffee in the urn, with a caffeine concentration of 100 mg per liter.
- Starting at time $t = 0$ the brewing machine is adding extra strong coffee, which has a concentration of 250 mg per liter. This new coffee is being added at a rate of 25 mL per minute.
- Starting at time $t = 0$ the student begins draining coffee out of the urn at a rate of 30 mL per minute.

Write down a differential equation which models the amount of caffeine in the urn as a function of time.

Exercise 1.2.5. In this problem we model the number of junk emails in Iva's inbox with a continuous function $J(t)$. Suppose the following:

- When the semester started ($t = 0$), Iva had 6000 emails in her inbox; 4000 of them were junk.
- Email is continuously flowing in to Iva's email inbox at a rate of 50 per day; 30% of the emails are junk.
- Each day, Iva randomly picks 20 emails to deal with[1] – once they have been dealt with, she moves them out of her inbox.

Write down a differential equation describing the number of junk emails in Iva's inbox.

Exercise 1.2.6. A big mixing vat contains 50 liters of a mixture in which the concentration of a certain chemical is 1.25 grams per liter. This mixture is being diluted by another mixture in which the concentration of the same chemical is 0.25 grams per liter. Each minute 6 liters of the less concentrated mixture are poured into the vat and 4 liters of the resulting new mixture are drained out. Construct a differential equation modeling this process.

1.3 Studying equations

Thus far we have learned what a differential equation is, and learned how to construct differential equations that model some simple population dynamics. Once we have such a differential equation, what do we do? If our differential equation describes the growth (or decline) of some population,

[1] This is not actually true, of course.

then perhaps we would like the results of our study to make some statement about the ultimate fate of that population. Or, perhaps we would like to "solve" the differential equation. In this section we give an overview of ways that we can mathematically study differential equations.

Before discussing differential equations, however, it is helpful to take a few minutes to discuss a topic we have past experience with: the study of algebraic equations.

Activity 1.3.1. *Find all solutions to the equation* $x^2 - 4 = 0$.

Activity 1.3.2. *Find all solutions to the system of equations*

$$\begin{aligned} x^2 + xy + x &= 0, \\ -y^2 + xy + 2y &= 0. \end{aligned}$$

Activity 1.3.3. *Find all solutions to the equation* $e^{-x} - x = 0$.

Activity 1.3.4. *Find all solutions to the equation* $e^{-x} + x = 0$.

You should find two solutions to the equation in Activity 1.3.1 and four solutions to the system in Activity 1.3.2. Activities 1.3.3 and 1.3.4 should cause you a bit more trouble, but that's OK – this is discussed in more detail below.

The number $x = -2$ is a solution to the equation in Activity 1.3.1. What does it mean, "is a solution"? It means that the number

$$(-2)^2 - 4,$$

which is obtained by replacing x by -2 on the left side of the equation, is equal to

$$0,$$

the right side of the equation.

Similarly, the pair $(x, y) = (-1, 0)$ is a solution to the system in Activity 1.3.2. This means that if we everywhere replace x by -1 and y by 0 then the *both* equations reduce to valid mathematical statements. Notice here that *a* solution is actually *two* objects (or, if you prefer, a single object with two parts).

In general, we say that a mathematical object is a ***solution*** to an equation if when we replace the unknown by that object the resulting mathematical statement is valid. For example, the number -2 is a solution to the equation in Activity 1.3.1 because

$$(-2)^2 = 4$$

is a valid mathematical statement.

Notice that the equation in Activity 1.3.1 has a second solution, namely 2. Thus in order for the problem of solving $x^2 - 4 = 0$ to a unique solution we must require that an additional condition hold true. For example, we might require that the solution be negative. In that case, there is indeed a unique solution.

Activities 1.3.3 and 1.3.4 contain two algebraic equations that could not be "solved" by simply manipulating them in order to isolate the unknown. Nevertheless, we can deduce that the equation in Activity 1.3.3 has exactly one solution, while the equation in Activity 1.3.4 does not have any solutions. One way to do this is to write the equations as

$$e^{-x} = x \quad \text{and} \quad e^{-x} = -x. \tag{1.3.1}$$

By plotting the curves $y = e^{-x}$, $y = x$, and $y = -x$, it is easy to see that the first equation has exactly one solution, while the second has none; see Figures 1.3.1 and 1.3.2. We can furthermore deduce that the solution to the first equation lies between zero and one. In fact, using a computer we can obtain a numerical approximation, finding that $x \approx 0.567$. (It is important to note that x is not equal to 0.567 – this is only an approximation!)

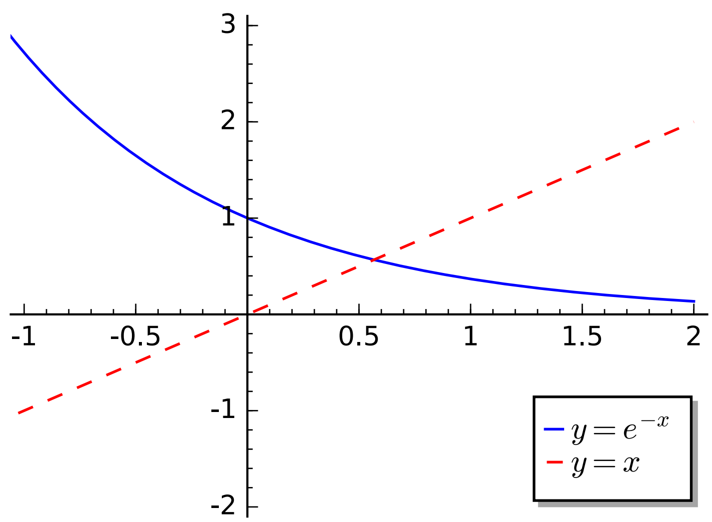

Figure 1.3.1: The curves $y = e^{-x}$ and $y = x$ cross at exactly one point. Thus the equation $e^{-x} = x$ has exactly one solution.

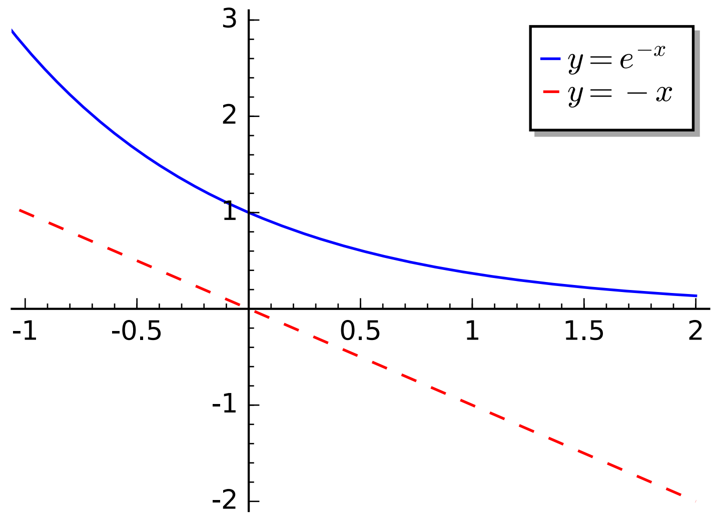

Figure 1.3.2: The curves $y = e^{-x}$ and $y = -x$ do not cross. Thus the equation $e^{-x} = -x$ does not have a solution.

The discussion of the algebraic equations (1.3.1) highlights several features of algebraic equations that are also important when studying differential equations:

- A given equation can might one solution, multiple solutions, or no solutions at all.
- In cases where there are multiple solutions, one can obtain a problem with a unique solution by imposing extra conditions.
- Often one can deduce that a solution to an equation exists without having an exact expression for the solution.
- In cases where we can deduce that solutions exist, but do not have exact expressions for the solutions, we can often deduce features of the solutions by carefully studying the equation.
- Often it is helpful to have a numerical approximation of the solution.

These two points are extremely important in the study of differential equations because "reasonable" differential equations have solutions, but it is very rare that there is a "nice" algebraic expression for these solutions. Thus our plan for studying differential equations has the following parts:

Existence theory This involves determining under what conditions a differential equation has a solution. In those cases where there exist multiple solutions, the existence theory tells us which extra conditions we can impose in order to obtain a problem for which there is a unique solution.

Qualitative study In most situations where our existence theory tells us that solutions to differential equations exist, the theory does not provide any direct information about those solutions. In particular, there is usually little hope that we can obtain a useful formula or expression for the solutions. Qualitative study is the art of deducing properties of solutions to differential equations without knowing or obtaining a formula or exact expression for the solutions.

Numerical study In situations where we cannot obtain an exact formula or expression for the solution to a differential equation, it can be very helpful to use a computer to obtain a numerical approximation. This is particularly useful for plotting (approximations of) solutions to differential equations.

In the next section, we begin our study of differential equations by presenting an existence theory. We conclude this section by discussing what it means for something to be a solution to a differential equation.

For algebraic equations, the unknowns are objects numbers (or collections of numbers), and solutions are those numbers that yield valid mathematical identities when they replace the unknowns in the equations. For differential equations, the unknowns are *functions*. Thus ***a solution to a differential equation*** is a function that, when the unknown is replaced by that function, the resulting identity is valid *as a statement about functions.*

Example 1.3.1. *Consider the differential equation*

$$\frac{du}{dt} = 7\cos(t).$$

The function $u(t) = 7\sin(t) + \sqrt{42}$ *is a solution to this equation because the* function

$$\frac{d}{dt}\left[7\sin(t) + \sqrt{42}\right]$$

is the same as the function

$$7\cos(t).$$

It is important to emphasize two aspects of the previous example:

1. The equality is in terms of functions, not numbers. The two sides cannot simply be equal for some values of t; they must agree for all admissible values of t.
2. When replacing u by the function $7\sin(t) + \sqrt{42}$, it is important to consider the left and right sides of the original equation independently. To do otherwise would be to assume that $7\sin(t) + \sqrt{42}$ is a solution.

Of course, the function $7\sin(t) + \sqrt{42}$ is not the only solution to the differential equation in Example 1.3.1. We know that any antiderivative of the function $7\cos(t)$ is a solution. Thus the general antiderivative

$$u(t) = \int 7\cos(t)\,dt = 7\sin(t) + C \tag{1.3.2}$$

is a solution for any value of the constant C. We call the function $u(t)$ in (1.3.2) the ***most general solution***. The concept of a "general solution" is discussed in greater detail below.‘

Activity 1.3.5. *Consider the differential equation*

$$\frac{dy}{dt} = 6y + 3t.$$

Show that the function

$$y(t) = e^{6t} - \frac{1}{12} - \frac{1}{2}t.$$

is a solution.

Exercise 1.3.1. Determine if the function $y(t) = 1 + 2e^t$ is a solution of the differential equation

$$\frac{dy}{dt} = (y-1)(y-2e^t).$$

Exercise 1.3.2.

1. Determine if the function $y(t) = e^t - 2t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.
2. Determine if the function $y(t) = 2e^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.
3. Determine for which values of the constant C the function $y(t) = Ce^t - t$ is a solution of the differential equation $\frac{dy}{dt} = y + t - 1$.

Exercise 1.3.3. Verify that the function $y(t) = \sqrt{\frac{4-t^3}{3t}}$ solves the differential equation

$$\frac{dy}{dt} = -\frac{t^2 + y^2}{2ty}.$$

Exercise 1.3.4. Consider the differential equation

$$\frac{dy}{dt} = \sqrt{y}.$$

Show that for any constant C, the function

$$y(t) = \left(\frac{1}{2}t + C\right)^2$$

is a solution (at least when $t \geq -2C$).

Exercise 1.3.5. Use integration to find the most general solution to the following differential equations.

1. $\frac{du}{dt} = \frac{1}{t}$
2. $\frac{du}{dt} = \cos t - t$
3. $\frac{du}{dt} = \ln t$

Exercise 1.3.6. For each of the differential equations in Exercise 1.3.5 find the solution such that $u(1) = 2$.

Exercise 1.3.7. In this problem we consider a differential equation that involves both first and second derivatives of the unknown function, namely

$$t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0.$$

Find those values of α such that the function $y(t) = t^\alpha$ solves the differential equation.

1.4 The initial value problem

In this section we discuss existence theory for differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{1.4.1}$$

where $f(t, y)$ is some function. Often we call the expression $f(t, y)$ the "right hand side" of the equation.

Example 1.4.1. *In Activity 1.3.5, the function f that determines the right hand side of the differential equation is*

$$f(t, y) = 6y + 3t.$$

This function f involves both the unknown function y and the independent variable t.

When the right hand side of the equation does not explicitly involve t, we write (1.4.1) as

$$\frac{dy}{dt} = f(y). \tag{1.4.2}$$

Example 1.4.2. *In the logistic growth model*

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

the function that determines the right side is

$$f(P) = rP\left(1 - \frac{P}{K}\right).$$

This function f only depends on the unknown function P and not the independent variable t.

Differential equations of the form (1.4.2), where the right hand side of the equation involves only the unknown function and not the independent variable, are called ***autonomous***.

In order to motivate the formulation of our existence theory, we now study three differential equations that take the form (1.4.2) and for which we are able to find exact formulas for solutions. They are:

$$\frac{dy}{dt} = r, \tag{1.4.3a}$$
$$\frac{dy}{dt} = ry, \tag{1.4.3b}$$
$$\frac{dy}{dt} = ry^2; \tag{1.4.3c}$$

in each case r is a numerical constant.

Example 1.4.3. *For any constant C, the function*

$$y(t) = C + rt$$

is a solution to (1.4.3a). *To see this we compute*

$$\frac{dy}{dt} = \frac{d}{dt}\left[C + rt\right] = r$$

This is the same function appearing on the right side of (1.4.3a), *thus the function* $y(t) = C + rt$ *is a solution.*

Since the constant C *is arbitrary, we actually obtain many solutions. The function* $y(t) = 27 + rt$ *is a solution, the function* $y(t) = -\pi + rt$ *is a solution, etc.*

Activity 1.4.1. *Show that for any constant* C *the function*

$$y(t) = Ce^{rt}$$

is a solution to (1.4.3b).

Activity 1.4.2. *Show that for any constant* C *the function*

$$y(t) = \frac{1}{C - rt}$$

is a solution to (1.4.3c).

Notice that each of the differential equations in (1.4.3) has a "family" of solutions, parametrized by the constant C. That is, there are an infinite number of solutions – one for each value of C. Plots of these families appear in Figures 1.4.1, 1.4.2, 1.4.3.

Activity 1.4.3. *The plots in Figures 1.4.1, 1.4.2, and 1.4.3 all assume* $r > 0$. *What do the plots look like when* $r < 0$*?*

Figures 1.4.1, 1.4.2, and 1.4.3 all share the following feature: the families of plots "fill the plane" in the sense that each point in the t–y plane lie on one, and only one, of the curves. Thus while each of the differential equations in (1.4.3) has an infinite number of solutions, we can specify exactly one solution by requiring that the graph of the solution pass through a particular point (t_0, y_0). This suggests the following:

> *Suppose we have a differential equation of the form* (1.4.1) *where* f *is some "reasonably nice" function. Then we expect there to be an entire family of solutions to the differential equation. If we require in addition that the graph of the solution* $y(t)$ *pass through a specified point* (t_0, y_0), *then we expect there to be only one solution.*

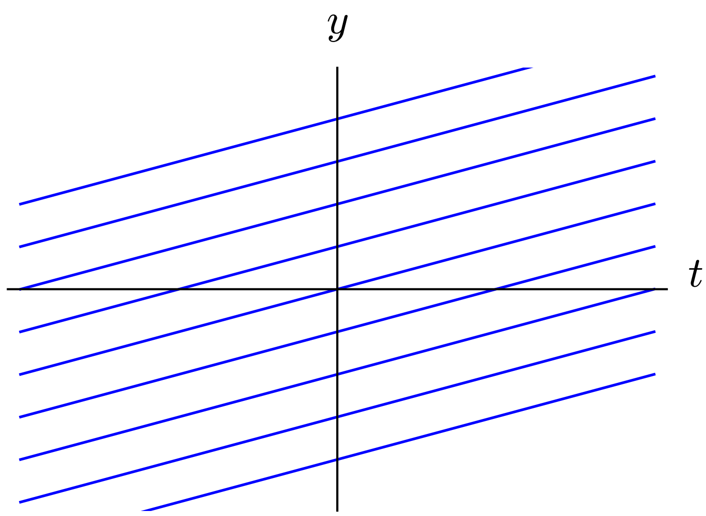

Figure 1.4.1: Plots of $y(t) = C + rt$ for various values of C, assuming $r > 0$.

The condition that the graph of $y(t)$ pass through the point (t_0, y_0) is called an ***initial condition***, and is equivalent to requiring $y(t_0) = y_0$. The problem of finding a function $y(t)$ satisfying both

$$\frac{dy}{dt} = f(t, y) \quad \text{and} \quad y(t_0) = y_0$$

is called the ***initial value problem (IVP)***. It is important to note that an IVP has two parts: the differential equation and the initial condition.

We can use the results of the activities above in order to solve the initial value problem associated to the differential equations in (1.4.3).

Example 1.4.4. *Consider the initial value problem*

$$\frac{dy}{dt} = y^2 \qquad y(1) = 7.$$

From Activity 1.4.2 we know that solutions to the differential equation take the form

$$y(t) = \frac{1}{C - t}.$$

The initial condition means that we must have

$$7 = \frac{1}{C - 1}.$$

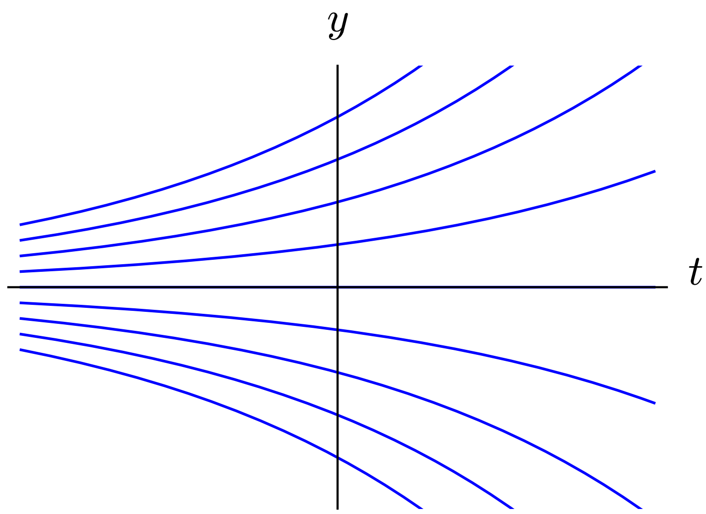

Figure 1.4.2: Plots of $y(t) = Ce^{rt}$ for various values of C, assuming $r > 0$.

Thus the constant C must be $C = 8/7$ and a solution to (1.4.5) *is*

$$y(t) = \frac{1}{\frac{8}{7} - t}.$$

Notice that the solution to this IVP does not exist for all values of t. The largest interval containing $t_0 = 1$ for which the solution is defined is $(-\infty, 8/7)$. Thus the solution can be extended arbitrarily far in to the "past" but cannot be extended very far in to the future.

Activity 1.4.4. *Solve the initial value problem*

$$\frac{dy}{dt} = 7 \qquad y(2) = 5.$$

Activity 1.4.5. *Find a solution to the following IVP:*

$$\frac{dy}{dt} = 3y \qquad y(0) = 6$$

Activity 1.4.6. *Find a solution to the following IVP:*

$$\frac{dy}{dt} = y^2 \qquad y(0) = 0.$$

(Hint: This is a trick question. What is the initial rate of change?)

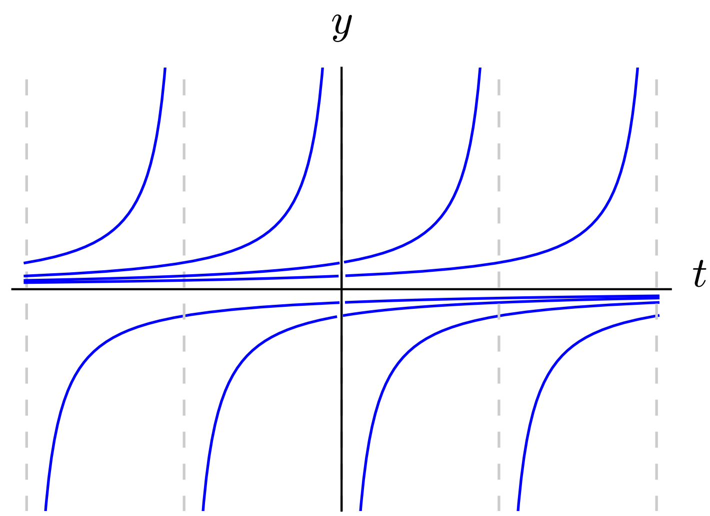

Figure 1.4.3: Plots of $y(t) = 1/(C - rt)$ for various values of C, assuming $r > 0$.

We now introduce the main theoretical tool in the study of ordinary differential equations, called the Fundamental Theorem of Ordinary Differential Equations (FTODE). Roughly speaking, it states that if the function f determining the right hand side of the differential equation is “nice enough” then the initial value problem has precisely one solution.

Theorem (FTODE)**.** *Consider the initial value problem*

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0. \tag{1.4.4}$$

If

- *the function $f(t, y)$ is continuous at the point (t_0, y_0) and*
- *the first derivatives (with respect to t and y) of $f(t, y)$ are continuous at the point (t_0, y_0),*

then there exists a function $y(t)$, defined on a small time interval containing t_0, satisfying (1.4.4).

A detailed proof of the FTODE is beyond the scope of this course. However, several important aspects of the proof are discussed in Section 2.7.

Example 1.4.5. *Consider the initial value problem*

$$\frac{dy}{dt} = y^2 \qquad y(1) = 7.$$

The function determining the right hand side is $f(y) = y^2$ *and the initial condition is* $(t_0, y_0) = (1, 7)$*.*

We compute

$$\frac{\partial f}{\partial t} = 0 \qquad \frac{\partial f}{\partial t} = 2y. \tag{1.4.5}$$

Since both of these functions are continuous at $(1, 7)$*, FTODE tells us that there is exactly one function* $y(t)$*, defined for values of* t *near* $t_0 = 1$*, satisfying* (1.4.5)*. In particular, the solution we found in Example 1.4.4 is the only solution!*

Example 1.4.6. *Consider the initial value problem*

$$\frac{dy}{dt} = 6y + 3t \qquad y(0) = \frac{11}{12}. \tag{1.4.6}$$

The function f *determining the right hand side is* $f(t, y) = 6y + 3t$ *(see Example 1.4.1). The point determining the initial condition is* $(t_0, y_0) = (0, 11/12)$*.*

We compute the partial derivatives

$$\frac{\partial f}{\partial t} = 3 \qquad \frac{\partial f}{\partial y} = 6.$$

Both of these are continuous at the point $(0, 11/12)$ *and thus we may apply FTODE to conclude that there is exactly one function* $y(t)$*, defined for values of* t *near* $t_0 = 0$*, satisfying* (1.4.6)*.*

(In fact, you know what the solution is! See Activity 1.3.5.)

Example 1.4.7. *Consider the initial value problem*

$$\frac{dy}{dt} = \frac{1}{y} \qquad y(2) = 0.$$

The function determining the right hand side is $f(y) = 1/y$ *and the initial condition is* $(t_0, y_0) = (2, 0)$*.*

We compute

$$\frac{\partial f}{\partial t} = 0 \qquad \frac{\partial f}{\partial y} = -\frac{1}{y^2}.$$

Since the derivative of f *with respect to* y *is not continuous (or even defined!) at* $(2, 0)$*, the FTODE does not tell us anything!*

Activity 1.4.7. *What does FTODE say about the following initial value problem?*

$$\frac{dy}{dt} = \sqrt{t^2 - y^2} \qquad y(1) = 0$$

Activity 1.4.8. *What does FTODE say about the following initial value problem?*

$$\frac{dy}{dt} = \sqrt{t^2 - y^2} \qquad y(1) = 1.$$

If the conditions of FTODE are satisfied by an initial value problem of the form (1.4.4), then we know that a unique solution $y(t)$ exists for times t in some interval containing the initial time t_0. This interval may not be infinite – it is possible the solution $y(t)$ cannot be uniquely continued past some time t_*. If this is the case, then we say that the solution has a ***singularity*** at time $t = t_*$. There are two types of singularities that can occur.

1. It might be that
$$\lim_{t \to t_*} |y(t)| = \infty. \tag{1.4.7}$$
This happens when the graph of $y(t)$ has a vertical asymptote at $t = t_*$. An example of this can be seen in Figure 1.4.3. When (1.4.7) occurs, we say that the solution $y(t)$ ***blows up*** at time t_* and call the singularity a ***blowup singularity***.

2. It might also be that as t approaches t_*, the solution $y(t)$ approaches a region where the function $f(t, y)$ in (1.4.4) and/or its derivatives are not continuous. A singularity of this type are called a ***Cauchy singularity***. An example of this phenomenon is explored in Exercise 1.4.5.

Notice that Cauchy singularities can only occur if the function $f(t, y)$ appearing in the differential equation does not satisfy the hypotheses of FTODE for some (t_*, y_*). If the function $f(t, y)$ and all of its derivatives are continuous for all points (t, y), then we say that f is ***globally smooth***.

Example 1.4.8. *The functions*

$$f(t, y) = t^2 - y^4, \quad f(t, y) = \frac{1}{1 + y^2}, \quad f(t, y) = t\cos(y)$$

are globally smooth. The functions

$$f(t, y) = \sqrt{t^2 - y^2}, \quad f(t, y) = \frac{1}{1 - y^2}, \quad ft, y = t\ln(y)$$

are not globally smooth.

Activity 1.4.9. *For each of the functions*

$$f(t,y)=\sqrt{t^2-y^2},\quad f(t,y)=\frac{1}{1-y^2},\quad f(t,y)=t\ln(y)$$

find a point (t_*, y_*) where the function is either not defined or not continuous (or where a derivative of the function is either not defined or not continuous).

It is important to note that even if $f(t, y)$ is globally smooth, solutions can still have blowup singularities. The classic example is the function $f(t, y) = y^2$; see Example 1.4.4 and Figure 1.4.3. However, it is also important to note that if $f(t, y)$ is globally smooth, then the only reason that a solution $y(t)$ to (1.4.4) might not be defined for all t is the possibility of a blowup. Thus: *If we can show that blowup cannot occur, then we know that the solution must exist for all times t.*

Exercise 1.4.1. Verify that for any constant C, the function

$$y(t)=Ce^{6t}-\frac{1}{12}-\frac{1}{2}t$$

satisfies the differential equation

$$\frac{dy}{dt}=6y+3t.$$

Use this to solve the IVP

$$\frac{dy}{dt}=6y+3t \qquad y(0)=10.$$

Exercise 1.4.2. Consider the IVP

$$\frac{dy}{dt}=y^3,\qquad y(0)=1.$$

1. Verify that the function

   $$y(t)=\sqrt{\frac{1}{1-2t}}$$

   solves this IVP.

2. Graph this solution.

3. State the domain on which this solution is defined.

4. Describe what happens to the solution as the independent variable approaches the endpoints of the domain. Why can't the solution be extended for more time?

Exercise 1.4.3. 1. Find the solution to the initial value problem

$$\frac{dy}{dt} = 1 - t^2, \qquad y(0) = 5.$$

2. Find the solution to the initial value problem

$$\frac{dy}{dt} = \cos t - \sin t, \qquad y(0) = \pi.$$

3. Suppose that $f(t)$ is some function and y_0 is a constant. Explain that the solution $y(t)$ to the initial value problem

$$\frac{dy}{dt} = f(t), \qquad y(0) = y_0$$

is given by

$$y(t) = y_0 + \int_0^t f(\tau)\, d\tau.$$

What's going on with the letter τ here?

Exercise 1.4.4. Consider the initial value problem

$$\frac{dy}{dt} = y^{1/2} \qquad y(0) = 0.$$

1. Show that $y(t) = t^2/4$ is a solution to the IVP.

2. Show that $y(0) = 0$ is a solution to the IVP.

3. This IVP has two solutions. Have we violated the part of FTODE that states there is only one solution? Explain.

Exercise 1.4.5. Verify that both $y_1(t) = 1$ and $y_2(t) = \cos(t)$ solve the IVP

$$\frac{dy}{dt} = -\sqrt{1 - y^2}, \qquad y(0) = 1$$

on the interval $[0, \frac{\pi}{2})$. Why does this not violate the Fundamental Theorem of ODE's?

1.5 First applications of FTODE

We begin with a graphical interpretation of FTODE. Suppose we have an initial value problem of the form

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0,$$

where the function f is globally smooth. Then FTODE states that for each t_0 and y_0 there is a unique function $y(t)$, defined for values of t nearby to t_0, such that the graph of the function passes through the point (t_0, y_0); see Figure 1.5.1.

(figure needed)

Figure 1.5.1: Graphical interpretation of FTODE.

The word "unique" in the statement of FTODE is very important. Not only does there exist a function passing through the point (t_0, y_0), but there is only one such function. Here is an important consequence of the uniqueness statement: Suppose we have two solutions to a differential equation. Then there graphs cannot cross at a point where the right hand side of the equation is globally smooth. I cannot emphasize this enough: *If the right hand side of a differential equation is globally smooth, then solutions to to that differential equations do not ever meet.*

The fact that solutions to nice differential equations don't cross turns out to be very handy for deducing properties of solutions to differential equations. To see this, consider the following examples.

Example 1.5.1. *Consider the differential equation*

$$\frac{dy}{dt} = y(1 - y).$$

It is easy to see that the function $f(y) = y(1-y)$ is globally smooth. Thus there are no points where solutions to this differential equation can cross.

It is also easy to verify that the function $y_*(t) = 0$ is a solution to the differential equation.

Suppose now that we have a solution to the initial value problem

$$\frac{dy}{dt} = y(1-y) \qquad y(0) = y_0,$$

where $y_0 > 0$.

At time $t = 0$ we have $y(0) > y_*(0)$. Furthermore, the solution $y(t)$ to this IVP cannot cross the solution $y_*(t) = 0$. Thus we must have $y(t) > y_*(t)$ for all t. In particular, we have $y(t) > 0$ for all t.

Thus we have shown that solutions to this IVP with positive initial condition are positive for all times.

Example 1.5.2. *Consider the differential equation*

$$\frac{dy}{dt} = y^2.$$

It is easy to verify that $f(y) = y^2$ is globally smooth. Thus there are no points where solutions to this differential equation can cross.

Recall from Activity 1.4.2 that $y_*(t) = \frac{1}{1-t}$ is a solution. The solution $y_*(t)$ satisfies $y_*(0) = 1$. Futhermore, $y_*(t)$ has a vertical asymptote at $t = 1$.

Consider now a solution $y(t)$ to the initial value problem

$$\frac{dy}{dt} = y^2 \qquad y(0) = y_0.$$

Let's consider two different scenarios.

- *If $y_0 < 1$, then it must be that $y(t) < y_*(t)$ for all t in the interval $[0, 1)$. In particular, the solution $y(t)$ is bounded above on that time interval.*

- *If, on the other hand, $y_0 > 1$, then it must be that $y(t) > y_*(t)$ for all t in the interval $[0, 1)$. This means that there must be some t_* in the interval $(0, 1]$ such that*

  $$\lim_{t \to t_*^-} y(t) = \infty.$$

  In other words, the asymptote of $y_*(t)$ "forces" the function $y(t)$ to also have an asymptote.

In the previous two examples we used our knowledge of one solution $y_*(t)$ to the differential equation in order to deduce properties of other solution $y(t)$. Notice that we did not need to obtain a formula for $y(t)$ in order to deduce properties about it. This is an example of the *qualitative analysis* mentioned in Section 1.3. In fact, the two examples above illustrate two important concepts in qualitative analysis: "equilibria" and "blowup."

In Example 1.5.1, the solution $y_*(t)$ we found was a constant function. Constant function solutions to differential equations are called ***equilibrium solutions***. In cases where the differential equation models the behavior of a population or physical system, equilibrium solutions correspond to populations or states that are unchanging; thus the name "equilibrium." Equilibrium solutions are important in qualitative analysis because they establish "barriers" that other solutions cannot cross.

Example 1.5.3. *Consider, as in Example 1.5.1, the differential equation*

$$\frac{dy}{dt} = y(1 - y). \tag{1.5.1}$$

It is easy to verify that there are two equilibrium solutions:

$$y_*(t) = 0 \quad \textit{and} \quad y_*(t) = 1.$$

Now suppose that $y(t)$ *is a solution to* (1.5.1) *with initial condition* $y(0) = y_0$*. The fact that solutions cannot cross allows us to make the following statements.*

- *If* $y_0 < 0$ *then* $y(t) < 0$ *for all times that* $y(t)$ *exists.*
- *If* $y_0 > 1$ *then* $y(t) > 1$ *for all times that* $y(t)$ *exists.*
- *If* $0 < y_0 < 1$ *then* $0 < y(t) < 1$ *for all times that* $y(t)$ *exists. In this case, we know that* $y(t)$ *cannot blow up and hence the solution must in fact exist for all times* t*.*

Activity 1.5.1. *Find the equilibrium solution(s) to the differential equation*

$$\frac{dy}{dt} = \frac{1 - y^2}{1 + y^2}.$$

Use the equilibrium solutions to make statements of the form

If $y(0)$ $\ldots$*, then* $y(t)$ $\ldots$*.*

Activity 1.5.2. *Find the equilibrium solution(s) to the differential equation*

$$\frac{dy}{dt} = y^2 - y\cos(t).$$

Use the equilibrium solutions to make statements of the form

If $y(0)$ ..., then $y(t)$ ....

Exercise 1.5.1. Here you study the differential equation

$$\frac{dy}{dt} = 1 - y^2.$$

1. Find the equilibrium solutions.

2. Suppose we have a solution $y(t)$ with initial condition $y(0) = y_0$ and $-1 < y_0 < 1$. What can you say about the long-time behavior of this solution?

Exercise 1.5.2. Show that $y_1(t) = 1$ and $y_2(t) = 1 + 2e^{-t}$ are solutions of the differential equation

$$\tfrac{dy}{dt} = (1 - y)(y - 2e^{-t}).$$

Based on the fact that $y_1(0) = 1$ and $y_2(0) = 3$ say something useful about solutions of the differential equation $\frac{dy}{dt} = (1 - y)(y - 2e^{-t})$ which obey the initial condition $1 < y(0) < 3$.

Exercise 1.5.3. Consider the initial value problem

$$\frac{dy}{dt} = -\frac{t^2 + y^2}{2ty}$$
$$y(1) = 1.$$

1. One can verify that

   $$y(t) = \sqrt{\frac{4 - t^3}{3t}}$$

   is a solution of this IVP. Use the Fundamental Theorem(s) of ODE's to explain why this function is the **only** solution of the IVP.

2. What is the domain of the solution discussed in the previous part of the problem?

3. Assuming that a solution $y(t)$ of the IVP

$$\frac{dy}{dt} = -\frac{t^2 + y^2}{2ty}$$
$$y(1) = 2$$

exists and is defined for all values of t with $0 < t < 2$ argue that $\lim_{t\to 0^+} y(t) = +\infty$.

Exercise 1.5.4.

1. Verify that the function

$$f(t) = \frac{2}{\sqrt[3]{8 - 7e^{3t}}}$$

is the unique solution of the IVP

$$\frac{dy}{dt} = y^4 - y, \quad y(0) = 2.$$

2. Explain why solutions $y(t)$ of

$$\frac{dy}{dt} = y^4 - y, \quad y(0) > 2$$

must satisfy

$$y(t) > \frac{2}{\sqrt[3]{8 - 7e^{3t}}}.$$

3. Explain why solutions $y(t)$ of

$$\frac{dy}{dt} = y^4 - y, \quad y(0) > 2$$

blow up in finite time.

Exercise 1.5.5. Find all equilibrium solutions of the logistic model with the harvesting term:

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10}\right) - 4.$$

Exercise 1.5.6. Find all equilibrium solutions to the equation

$$\frac{dy}{dt} = y(1 + y)(y - e^t).$$

Exercise 1.5.7. Find all equilibrium solutions to the equation

$$\frac{dy}{dt} = y^2 + y + y \sin t + \sin t.$$

Exercise 1.5.8. Consider the differential equation

$$\frac{dy}{dt} = y^2 - a,$$

where a is some constant. For which values of a does the equation have equilibrium solutions?

Writing assignment 1: Equilibrium solutions for the logistic model

In this assignment you analyze the equilibrium solutions to the logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{1.5.2}$$

and to the modified logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) + h, \tag{1.5.3}$$

where h is a constant.

Your report is to address the following:

- Introduce the equations by discussing the assumptions that lead to the models (1.5.2) and (1.5.3).
- Determine for which values of the parameters r, K, and h (in the case of the modified equation) the equations admit equilibrium solutions.
- Interpret your findings in the context of population modeling.