Review of integration techniques

Change of variables

Recall that the basic idea is to reverse engineer the chain rule. For example, if we let $u = \tau^2 + 1$, then $du = u'(\tau)\, d\tau = 2\tau\, d\tau$. Thus

$$\int_0^t \underbrace{e^{\tau^2+1}}_{e^u} \underbrace{2\tau\, d\tau}_{u'(\tau)\, d\tau} = \int_{(t)^2+1}^{(1)^2+1} e^u\, du = [e^u]_1^{t^2+1} = e^{t^2+1} - e^1.$$

We can do the same thing if the integrand involves a function y. For example, suppose we let $u = y(\tau) + 5$. Then

$$\int_0^t \underbrace{\frac{1}{y(\tau)+5}}_{\frac{1}{u}} \underbrace{\frac{dy}{d\tau}\, d\tau}_{u'(\tau)\, d\tau} = \int_{y(0)}^{y(t)} \frac{1}{u}\, du = [\ln(u)]_{y(0)}^{y(t)} = \ln(y(t)) - \ln(y(0)) = \ln\left(\frac{y(t)}{y(0)}\right).$$

Now you try it! Compute

$$\int_0^t \frac{\tau^2+1}{2} \tau\, d\tau \qquad \text{and} \qquad \int_0^t \cos(y(\tau))\, \frac{dy}{d\tau}\, d\tau.$$

Partial fractions

The idea is that if the denominator of a fraction can be factored, then there is a way to rewrite the fraction in a manner more convenient for integration. For example, suppose we want to write

$$\frac{1}{(y-2)(y+3)} = \frac{A}{y-2} + \frac{B}{y+3}.$$

If we multiply both side by $(y-2)(y+3)$, we see that we need to have $1 = A(y+3) + B(y-2)$. We rewrite this as

$$(0)y + (1) = (A+B)y + (3A - 2B).$$

Thus we need

$$A + B = 0 \qquad \text{and} \qquad 3A - 2B = 1.$$

This is two equations and two unknowns. We find that $A = 1/5$ and $B = -1/5$. Thus

$$\frac{1}{(y-2)(y+3)} = \frac{\frac{1}{5}}{y-2} + \frac{-\frac{1}{5}}{y+3}.$$

You try it! Use partial fractions to rewrite

$$\frac{1}{(y+5)(y-4)}.$$

The compute

$$\int_0^t \frac{1}{(y+5)(y-4)} \frac{dy}{d\tau}\, d\tau.$$

Preparing for Exam 1

Problem 1. (10 minutes)

Solve the initial value problem

$$\frac{dy}{dt} = y^3 \qquad y(0) = 2.$$

Problem 2. (20 minutes)

Consider a 200 L tank, which is initially full of salt water. The water in the tank is being drained at a rate of 15 L/min. At the same time, a solution with concentration 12 g/L is entering the tank at a rate of 10 L/min. During this process, the tank is being perfectly mixed.

A. Write down a differential equation which describes how the amount of salt in the tank changes in time.

B. Find the propagator function associated to your differential equation.

C. Suppose that initially there are 100 g of salt in the tank. Find an expression for the amount of salt in the tank at time π. You do not need to simply your expression.

Problem 3. (10 minutes)

In this problem you study the differential equation

$$\frac{dP}{dt} = P^3 - 4P^2 + 4P.$$

A. Make a sketch of the slope field for differential equation.

B. Determine the stability of each of the equilibrium points. Explain your reasoning.

C. Based on your slope field, describe the long term behavior of the solution to the initial value problem with initial condition $P(0) = 1$.