4.5 Initial boundary value problems

We conclude this chapter by describing some mathematical problems that are related to (4.13). First, we introduce some vocabulary.

A ***partial differential equation (PDE)*** is an equation where the unknown is a function of multiple variables, and where the equation involves partial derivatives with respect to these variables. The ***order*** of a differential equation is the number of derivatives involved.

example:1D-pde

Example 4.8. *We list the following examples of partial differential equations:*

1. *The one-dimensional wave equation is*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

   where the unknown u is a function of t and x. This PDE is second order in both t and x.

2. *The one-dimensional heat equation is*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

   where the unknown u is a function of t and x. This PDE is first order in t and second order in x.

3. *The two-dimensional Laplace equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

   where the unknown u is a function of x and y. This PDE is second order in both x and y.

In the wave and heat equations in Example 4.8, the variable t refers to time. If one of the variables in a PDE represents time, the equation is called an ***evolution***

equation. Such equations typically describe how some quantity evolves in time. The Laplace equation is not an evolution equation.

For evolution equations, it is natural to specify some "initial state" of the unknown, and then to try to find a solution to the differential to the PDE such that the solution agrees with the initial state at the initial time. Our experience studying ordinary differential equations (ODEs) leads us to the following:

- If a PDE is first order in time, then we expect to be able to specify the initial value of the unknown.

- If a PDE is second order in time, then we expect to be able to specify both the initial value and initial "velocity" (i.e. the first time derivative) of the unknown.

For the wave equation, the initial value and initial velocity are given by functions s and v, respectively. Together, the function s and v are called the ***initial conditions (IC)***. (In the case of the heat equation, only the initial value s is present in the initial condition.)

If we are given function s and v, then we can try to find a function u satisfying

$$\begin{aligned} &\text{PDE:} && \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \\ &\text{IC:} && u(0,x) = s(x) \quad \frac{\partial u}{\partial t}(0,x) = v(x). \end{aligned} \tag{4.15}$$

1D-wave-IVP

The problem (4.15) is called the ***initial value problem*** for the one-dimensional wave equation.

Exercise 4.9. *What form does the initial value problem for the one-dimensional heat equation take?*

The initial value problem (4.15) does not specify what values of x are being considered. The set of x values being considered is called the ***spatial domain***. If the spatial domain is not specified, then it is assumed to be as large as possible. Thus for (4.15) we assume that the spatial domain is $\mathbb{R}$.

Frequently, as in the case of (4.13), we restrict the spatial domain to some finite region. In this case, it is natural to impose conditions on the unknown u along the boundary of the spatial domain; such conditions are called ***boundary conditions (BC)***. An example of a boundary condition is the Dirichlet condition (4.3).

Here are two more examples.

Example 4.10. *Suppose the spatial domain of a PDE is the interval* $[-L, L]$*.*

1. *The function u satisfies the* **Neumann boundary condition** *if*

$$\left.\frac{\partial u}{\partial x}\right|_{x=-L} = 0 \quad \textit{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

2. *The function u satisfies the* **periodic boundary condition** *if*

$$u|_{x=-L} = u|_{x=L} \quad \textit{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L}.$$

These boundary conditions are explored further in subsequent chapters.

The problem of finding a solution to an evolution PDE having specified initial conditions and specified boundary conditions is called the ***initial boundary value problem (IBVP)***. The problem (4.13) is an example; here is another.

Example 4.11. *The Neumann IBVP for the wave equation on the spatial domain* $[-L, L]$ *takes the form*

$$\begin{aligned}
&\textit{PDE:} && \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \\
&\textit{BC:} && \frac{\partial u}{\partial x}(t, -L) = 0, \quad \frac{\partial u}{\partial x}(t, L) = 0 \\
&\textit{IC:} && u(0, x) = s(x), \quad \frac{\partial u}{\partial t}(0, x) = v(x).
\end{aligned}$$

Exercise 4.12. *What form does the periodic initial boundary value problem for the wave equation on spatial domain* $[-L, L]$ *take?*