4.4 The initial value problem and the Fourier series hypothesis

rier-series-hypothesis

By taking combinations of the standing wave solutions we can construct large numbers of solutions to the wave equation. In the case of the simple harmonic oscillator, we can in fact build all solutions by taking combinations of scaling solutions. This raises the question: Can every solution to the wave equation be constructed from standing wave solutions?

To investigate this question, we take a closer look at those solutions that can be constructed from standing waves. Since these solutions are finite combinations of standing wave solutions, we can write a generic combination as

$$u_N(t,x) = \sum_{k=1}^{N} \Big(\alpha_k \cos(\omega_k t)\psi_k(x) + \beta_k \sin(\omega_k t)\psi_k(x)\Big) \tag{4.12}$$

1D-first-partial-sum

for some constants α_k and β_k.

When studying ordinary differential equations, we determined the constants appearing in “general solutions” using initial conditions. For a second-order ordinary differential equation, the initial value problem specified the initial position and velocity. This motivates us to try a similar approach for the wave equation.

The appropriate version of the initial value problem for the vibrating string is the ***one-dimensional Dirichlet initial boundary value problem (IBVP)***, which seeks to find a function u satisfying the following:

$$\begin{aligned}
&\text{one-dimensional wave equation:} && \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\\
&\text{Dirichlet boundary condition:} && u(t,-L) = 0, \quad u(t,L) = 0,\\
&\text{initial conditions:} && u(0,x) = s(x), \quad \frac{\partial u}{\partial t}(0,x) = v(x),
\end{aligned} \tag{4.13}$$

1D-Dirichlet-IBVP

where the function $s(x)$ represents the initial shape of the wave and the function $v(x)$ represents the initial velocity.

By construction, the function $u_N(t, x)$ given by (4.12) already satisfies the wave equation and the Dirichlet boundary condition. We also compute that the shape of the solution at $t = 0$ is given by

$$u_N(0, x) = \sum_{k=1}^{N} \alpha_k \psi_k(x),$$

while the velocity at $t = 0$ is given by

$$\frac{\partial u_N}{\partial t}(0, x) = \sum_{k=1}^{N} \beta_k \omega_k \psi_k(x).$$

Thus the initial conditions in the IBVP are satisfied only if

$$\begin{aligned} s(x) &= \sum_{k=1}^{N} \alpha_k \psi_k(x) \\ v(x) &= \sum_{k=1}^{N} \beta_k \omega_k \psi_k(x). \end{aligned} \tag{4.14}$$

1D-construct-IC

Hence we see that initial conditions determine the constants in (4.12) if and only if there is a unique way to express the functions s and v as the sum of a finite number of the functions ψ_k.

Important Point 4.7. *The ability to solve an initial boundary value problem using standing wave solutions is equivalent to being able to construct an arbitrary function s out of the functions ψ_k.*

Unfortunately, there are many functions $s(x)$ that cannot be expressed as a finite sum of the functions $\psi_k(x)$. A good example is the function $s(x) = L - |x|$, which represents an initial shape of a string that has been pinched in the middle and pulled up.

At first, this may seem to derail our plan to construct solutions to the initial boundary value problem using sums of standing wave solutions. However, while the function $s(x) = L - |x|$ cannot be *exactly* expressed as the finite sum of the functions $\psi_k(x)$, it can be *approximated* by such a sum. It is easy to verify that by choosing

$$\alpha_k = \begin{cases} 2\left(\frac{2L}{k\pi}\right)^2 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

then the function

$$\sum_{k=1}^{N} \alpha_k \psi_k(x)$$

very closely approximates $s(x)$; see Figure 4.1. (Notice that this sum only involves cosines. For example, we have

$$\sum_{k=1}^{7} \alpha_k \psi_k(x) = 2\left(\frac{2L}{\pi}\right)^2 \cos\left(\frac{\pi x}{2L}\right) + 2\left(\frac{2L}{3\pi}\right)^2 \cos\left(\frac{3\pi x}{2L}\right) + 2\left(\frac{2L}{5\pi}\right)^2 \cos\left(\frac{5\pi x}{2L}\right) + 2\left(\frac{2L}{7\pi}\right)^2 \cos\left(\frac{7\pi x}{2L}\right).$$

This is because $s(x)$ is an even function, and $\psi_k(x)$ is even only when k is odd.)

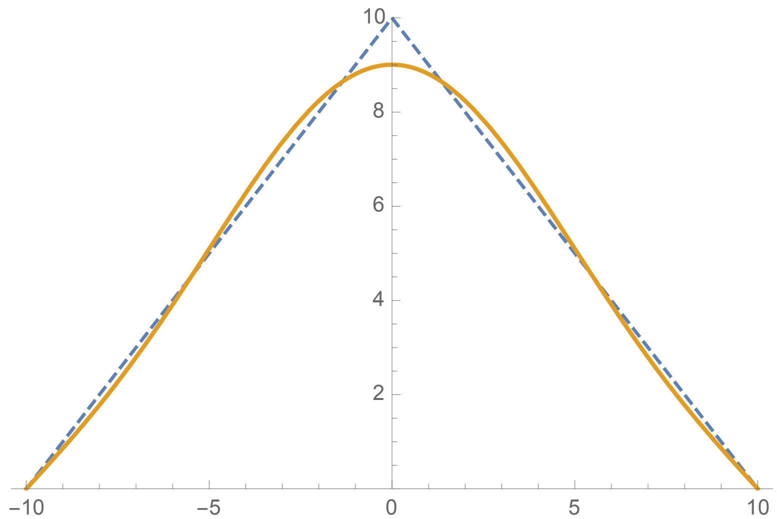

Figure 4.1: Plots of the function $s(x) = L - |x|$ (dashed) and the sum $\sum_{k=0}^{7} \alpha_k \psi_k(x)$ (solid) with $L = 10$. Even though the sum consists of only four cosine functions, it is a reasonable approximation of the function $s(x)$.

Figure:first-fourier-pl

In 1822, Joseph Fourier published his work *Théorie analytique de la chaleur*, in which he claimed that any function can be approximated by a suitable sum of trigonometric functions. . . and that if infinite sums are allowed, then the sum is in fact equal to the function. In other words, he proposed constructing functions from an infinite series of trigonometric functions, much in the same way that many functions can be constructed from a power series consisting of an infinite sum of polynomials. I call this claim the "Fourier series hypothesis."

Fourier's claim is not quite true in the way that he stated it in 1822, in large part due to the fact that many ideas in calculus were not well formulated at the time. (In fact, as you learn in the real analysis course, Fourier's claim inspired the development of much of the modern theory that puts calculus on a more solid foundation.) However, his hypothesis is "essentially true" and is one of the most important ideas in applied mathematics today.

In the Part II of this course we investigate in much more detail the sense in which the Fourier series hypothesis is true. In particular, we address the following questions:

Best approximation problem: Suppose we have an infinite collection of functions $\psi_1, \psi_2, \psi_3, \ldots, \psi_N$ and we have another function s. How should we choose the numbers $\alpha_1, \alpha_2, \alpha_3$ so that the function

$$s_N = \sum_{k=1}^{N} \alpha_k \psi_k$$

best approximates s?

Convergence: Suppose we have addressed the best approximation question above. Can we show that, in the limit as $N \to \infty$, the function s_N above approaches the original function s?

For now, let us just observe that if the Fourier series hypothesis is true, then we can use standing waves in order to solve the initial boundary value problem for the wave equation.