Part 1

The simple harmonic oscillator and the wave equation

In the first part of the course we revisit the simple harmonic oscillator, previously discussed in differential equations class. We use the discussion here to remind ourselves of the "eigenstuff" program for solving linear equations. We then make a first derivation of the one dimensional wave equation and to ask some questions that arise when we use the eigenstuff method to study waves.

CHAPTER 1

Review: The simple harmonic oscillator

Recall the simple harmonic oscillator

$$m\frac{d^2u}{dt^2} = -ku. \tag{1.1}$$

Here u represents the displacement from equilibrium of some oscillator, and (1.1) is simply Newton's formula $ma = F$ with the force being given by Hooke's formula $F = -ku$. It is common to set $\omega^2 = k/m$, so that (1.1) becomes

$$\frac{d^2u}{dt^2} = -\omega^2 u. \tag{1.2}$$

Introducing $v = \frac{du}{dt}$ we can write (1.2) as the first-order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.3}$$

Let

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0, \end{pmatrix} \tag{1.4}$$

and look for solutions to (1.3) of the form

$$\mathbf{u} = A(t)\,\mathbf{u}_*, \tag{1.5}$$

where $\mathbf{u}_* = \begin{pmatrix} u_* \\ v_* \end{pmatrix}$ is a constant vector. Solutions of the form (1.5) are called ***scaling solutions*** because they are simply rescaled as time progresses.

Plugging (1.5) in to (1.3) we see that in order to be a solution we must have

$$\frac{1}{A}\frac{dA}{dt}\mathbf{u}_* = M\mathbf{u}_*.$$

Since the right side of this equation is time independent, there must be some constant λ such that

$$\frac{1}{A}\frac{dA}{dt} = \lambda \quad \text{and} \quad M\mathbf{u}_* = \lambda\mathbf{u}_*. \tag{1.6}$$

Notice that the second equation is the equation for the *eigenvalue problem*.

Important Point 1.1. *Scaling solutions take the form* $A(t)\mathbf{u}_*$ *where*

- $\mathbf{u}_*$ *is an eigenvector of* M *and*
- *the amplitude function* $A(t)$ *satisfies some differential equation depending on the corresponding eigenvalue* λ.

The matrix M appearing in (1.4) has eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$. The corresponding scaling solutions are

$$\mathbf{u}_1 = e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix} \tag{1.7}$$

Using linearity (see Exercise 1.3) we see that

$$\mathbf{u} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 \tag{1.8}$$

is a solution to (1.3) for any constants α_1 and α_2. This is, in fact, a ***general solution*** to (1.3), meaning that any solution can be obtained by appropriately choosing α_1 and α_2; see Exercise 1.1.

Important Point 1.2. *If an equation is linear, then one can obtain more solutions by using linearity to combining scaling multiple solutions. If any solution can be constructed this way, then we say that the list of scaling solutions is complete. The completeness of the scaling solutions is equivalent to the completeness of corresponding list of eigenvectors.*

You might be concerned that the solutions in (1.7) are complex-valued, while the differential equation (1.3) only involves real numbers. We can, however, obtain real solutions by careful choices of α_1 and α_2. To do this we use the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.9}$$

which implies that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

For example, if we choose $\alpha_1 = 1$ and $\alpha_2 = 1$, then we have

$$\mathbf{u} = (e^{i\omega t} + e^{-i\omega t})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos\omega t \\ 0 \end{pmatrix}.$$

This issue is explored in more detail in Exercise 1.2. For now, we note the following.

Important Point 1.3. *If we obtain complex-valued scaling solutions we can use linearity to find real-valued solutions by carefully choosing the constants in the general solution and using the Euler identity* (1.9).

Exercises

Exercise 1.1.

(1) Show that for any vector $\mathbf{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ we can choose α_1 and α_2 such that

$$\alpha_1 \begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

(2) Use the previous part to show that for any initial value $\mathbf{u}_0$ the initial value problem

$$\frac{d}{dt}\mathbf{u} = M\mathbf{u} \qquad \mathbf{u}(0) = \mathbf{u}_0$$

has a solution of the form (1.8).

(3) Explain how this shows that all solutions to (1.3) take the form (1.8).

Exercise 1.2. The scaling solutions given in (1.7) involve complex numbers. Here we see how these nevertheless give rise to real-valued solutions.

(1) Find the solution $\mathbf{u}_c$ to the initial value problem

$$\frac{d}{dt}\mathbf{u} = M\mathbf{u} \qquad \mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the Euler identity (1.9) to show that $\mathbf{u}_c$ is in fact real-valued.

(2) Find the solution $\mathbf{u}_s$ to the initial value problem

$$\frac{d}{dt}\mathbf{u}_s = M\mathbf{u}_s \qquad \mathbf{u}_s(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{u}_s$ is also fact real-valued.

(3) Explain why

$$\mathbf{u} = \beta_c\mathbf{u}_c + \beta_s\mathbf{u}_s$$

is a general solution to (1.3), and that real initial conditions give rise to real-valued solutions.

Exercise 1.3. A differential equation is called ***linear*** if the following property holds:

- If u and v are solutions and α, β are numbers, then $\alpha u + \beta v$ is also a solution.

(This property is sometimes called the "superposition" principle.) Show that (1.1) and (1.3) are linear differential equations.

CHAPTER 2

Power series solutions to differential equations

Here we address the issue of existence of solutions to linear ordinary differential equations, such as the differential equation that determined the amplitude functions in (1.7). The following theorem was stated in differential equations class.

Theorem 1 (Fundamental theorem of ordinary differential equations). *Consider the initial value problem*

$$\frac{dy}{dt} = F(t, y) \qquad y(t_0) = y_0.$$

If the function F is continuous, and has continuous partial derivatives, in a neighborhood of (t_0, y_0) then there exists a unique solution $y(t)$ to the initial value problem, defined for t in some time interval containing t_0.

The proof of Theorem 1 (which is a version of the Picard Theorem) belongs to the real analysis course. In this class we make use of an approach that yields solutions to certain types of linear equations: power series.

We say that a function $f(x)$ can be ***represented by a power series*** (centered at zero) if there exists constants $a_0, a_1, \ldots$ and constant $R > 0$ such that for $|x| < R$ the sum

$$\sum_{k=0}^{\infty} a_k x^k, \tag{2.1}$$

converges to $f(x)$. In other words

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{for } |x| < R. \tag{2.2}$$

We want the function with a power series representation (2.2) to be differentiable. We can ensure this by requiring that the series converge absolutely on the interval of convergence. The following is proved rigorously in a real analysis course.

Theorem 2 (Properties of power series).

(1) *(Uniqueness) Suppose that both*

$$\sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k x^k$$

converge absolutely to $f(x)$ for $|x| < R$. Then $a_k = b_k$ for each k.

(2) *(Differentiablity) Suppose that*

$$\sum_{k=0}^{\infty} a_k x^k$$

converges absolutely to $f(x)$ for $|x| < R$. Then $f(x)$ is differentiable on the interval $|x| < R$ and

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}.$$

Let's now see how to use power series representations in order to solve differential equations. Consider the equation

$$\frac{dA}{dt} = \lambda A \tag{2.3}$$

appearing in Chapter 1; here λ is some fixed number. (Of course, we "know" from our differential equations course that A should be a multiple of $e^{\lambda t}$, but – for the sake of the discussion – let's temporarily put that "knowledge" on hold.)

Let's also assume that A has a power series representation

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad |t| < R.$$

Inserting this representation in to (2.3) we find that

$$\sum_{k=0}^{\infty} k a_k t^{k-1} = \sum_{k=0}^{\infty} \lambda a_k t^k.$$

We rearrange this to obtain

$$\sum_{k=0}^{\infty} [(k+1)a_{k+1} - \lambda a_k]\, t^k = 0. \tag{2.4}$$

It is important to note that in this last equation the equal sign is equality *as functions*. In other words, (2.4) must hold for all $|t| < R$.

However, the power series for the zero function has all coefficients equal to zero. Thus if (2.4) holds, then we must have

$$(k+1)a_{k+1} - \lambda a_k = 0 \qquad \text{for } k = 0, 1, 2, 3 \dots. \tag{2.5}$$

Thus the differential equation (2.3) gives rise to a *recurrence relation* that must be satisfied by the coefficients a_k.

The recurrence relation (2.5) means that

$$\begin{aligned} a_0 &= \text{anything} \\ a_1 &= \lambda a_0 \\ a_2 &= \frac{1}{2}\lambda a_1 = \frac{1}{2}\lambda^2 a_0 \\ a_3 &= \frac{1}{3}\lambda a_2 = \frac{1}{3\cdot 2}\lambda^3 a_0 \\ &\vdots \\ a_k &= \frac{1}{k!}\lambda^k a_0. \end{aligned}$$

Thus we see that $A(t)$ has power series representation

$$A(t) = a_0 \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda t)^k.$$

The sum is (of course!) the well-known power series expression for $e^{\lambda t}$, which converges for all t (and is what we already "knew"). The constant a_0 can be chosen freely, representing our choice of initial condition for (2.3). Since the process of forming a general solution involves multiplying each scaling solution by an arbitrary constant, we are free to choose a_0 in a manner that is convenient – we'll choose $a_0 = 1$.

We summarize this discussion as follows.

Important Point 2.1 (Power series solutions). *Solutions to (certain) linear ordinary differential equations can be found by looking for power series representations. Inserting a generic power series in to the differential equation yields a recurrence relation for the coefficients. Some of the coefficients can be chosen freely (corresponding to freedom of choosing initial conditions); the remaining coefficients are determined by the relation. It is important to make sure that the resulting power series actually converges absolutely.*

Both the power series solution to (2.3) and the eigenvalue problem in Chapter 1 involve free parameters. As discussed above, the free choice for the power series solution corresponds to free choice of initial conditions for the differential equations. The free choice for the eigenvalue problem is associated to the fact that for any given eigenvalue the corresponding eigenvectors form a vector space. Thus one may multiply an eigenvector by any constant and obtain another eigenvector. The consequence of all this is that our scaling solutions can be multiplied by any constant and still be a scaling solution. At the end of the day, we multiply these scaling solutions by yet another constant when forming the general solution. Thus it really doesn't matter what choice of constant we choose when constructing the power series or when finding eigenvectors. When possible we make "strategic" choices; in some cases there are "industry standards" to follow.

Important Point 2.2 (Free constants). *The power series solutions and eigenvectors that form scaling solutions both involve freely chosen constants. The choice of these constants does not matter for the purposes of constructing a general solution. In many cases we follow conventional choices; in other cases we make strategic choices.*

Exercises

Exercise 2.1. In calculus 2, we learned how to construct a power series representation for a previously known function; we called this *Taylor series*. Write down the Taylor series expansions for the following functions.

(1) e^x

(2) $\ln(1-x)$ (Hint: What is the series for $1/(1-x)$?)

(3) $\cos x$

(4) $\sin x$

Note: In calculus 1 and 2 these functions were defined in some sort of ad hoc manner. A more sophisticated point of view would be to *define* the functions by their power series expansions.

Exercise 2.2. Show that the series

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k+1}$$

converges absolutely for $|x| < 1$. Then show that the series converges to

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Exercise 2.3.

(1) Recall that a function $f(x)$ is called ***even*** if $f(-x) = f(x)$ for all x. Show that the power series representation of an even function contains only even powers of x. [Hint: Use the uniqueness property of power series.]

(2) A function $f(x)$ is called ***odd*** if $f(-x) = -f(x)$ for all x. Show that the power series representation of an odd function contains only odd powers of x.

(3) Any function can be written as the sum of an even function and an odd function. To see this, note that

$$f(x) = \frac{1}{2}\underbrace{[f(x) + f(-x)]}_{\text{even}} + \frac{1}{2}\underbrace{[f(x) - f(-x)]}_{\text{odd}}.$$

Give an alternate "proof" using power series.

Exercise 2.4. Use the power series method to construct the most general solution to

$$f'(x) = (1 + x)f(x).$$

Exercise 2.5. Use the power series method to construct the most general solution to

$$f'(x) = x\,f(x).$$

Does the series look familiar?

Exercise 2.6. One can also use the power series method to construct solutions to second order differential equations. As an example, consider (1.2). In this case, initial conditions correspond to specifying $u(0) = u_0$ and $u'(0) = v_0$.

(1) Assume that solution u to (1.2) has power series representation $\sum_{k=0}^{\infty} a_k t^k$. Insert this in to (1.2) and obtain a recursion relation for the coefficients.

(2) Your recursion relation should leave a_0 and a_1 fixed, and should "split" over the even/odd-numbered coefficients in the sense that a_0 depends on a_2, a_4, etc. and a_1 determines a_3, a_5, etc. Rewrite your recursion relations to account for this splitting by finding formulas of the form

$$a_{2k} = \boxed{\text{stuff involving } a_{2k-2}} \quad \text{and} \quad a_{2k+1} = \boxed{\text{stuff involving } a_{2k-1}}$$

(3) Find an even solution to (1.2) by setting $a_1 = 0$ and $a_0 = 1$. The resulting power series should look very familiar!

(4) Find the odd solution to (1.2) by setting $a_0 = 0$ and $a_1 = 1$. Again, the result should look very familiar!

(5) Use linearity to construct the general solution to (1.2).

(6) From (1.7) we see that $e^{i\omega t}$ is a solution to (1.2). What choices of a_0 and a_1 gives rise to this solution?

Exercise 2.7. (Optional) It is important to be clear about the logic behind solving differential equations with power series. Our method is to go searching for solutions that admit power series representations. Thus if the equation has solutions, and if those solutions have power series representations, our method will find it. (Furthermore, if we find a solution using the power series method, then by Theorem 1 it is *the* unique solution.)

However, not all "nice" functions admit power series representations. To explore this issue further, we say that a function $f(x)$ is ***real analytic*** for $|x| < R$ if it has a power series representation that converges absolutely for $|x| < R$. We say that $f(x)$ is ***smooth*** for $|x| < R$ if f can be differentiated any number of times. (Since differentiable functions are continuous, this means that all derivatives of smooth functions are continuous.) The symbol $C^\infty(-R, R)$ is used to represent the collection of smooth functions defined for $|x| < R$, and the symbol $C^\omega(-R, R)$ is used to represent the collection of real analytic functions.

In this problem we show that the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{else} \end{cases}$$

is smooth, but not real analytic, on any interval $|x| < R$.

(1) Show that f is continuous by showing that

$$\lim_{x \to 0} e^{-1/x^2} = 0.$$

(2) Show that f is smooth by arguing that

$$\lim_{x \to 0} \left(\frac{d^k}{dx^k} \left[e^{-1/x^2} \right] \right) = 0, \qquad k = 1, 2, 3, \ldots$$

(3) Show that $f^{(k)}(0) = 0$ for all k and thus that the Taylor series for f is the same as the Taylor series for the zero function.

(4) Notice that $f(x)$ is not the zero function, and conclude that $f \notin C^\omega(-R, R)$.

(5) Finally, recall that

$$e^y = 1 + y + \frac{1}{2}y^2 + \frac{1}{3!}y^3 + \ldots.$$

Use this to find a series representation for f in terms of negative powers of x. Such a series is called a *Laurent series*. There is a nice discussion on the relevant Wikipedia page: https://en.wikipedia.org/wiki/Laurent_series.

CHAPTER 3

From the simple harmonic oscillator to the wave equation

As presented in Chapter 1, the simple harmonic oscillator (1.1) arises by combining Newton's formula $F = ma$ with Hooke's statement that the force of a spring is proportional to the displacement. (The proportionality constant k is sometimes called the *spring constant*.)

One can also obtain (1.1) by thinking about energy[1]. Let $K = \frac{1}{2}m\left(\frac{du}{dt}\right)^2$ be the *kinetic energy* of a particle having mass m and with u describing the location of its one dimensional motion. To any particular location, we associate a corresponding *potential energy* $V = V(u)$. The total energy E is the sum of the kinetic and potential energy:

$$E = K + V = \frac{1}{2}m\left(\frac{du}{dt}\right)^2 + V(u).$$

Notice that E is a function of u (and its derivative) and is thus a function of time. We now enforce the condition that E be conserved; that is, we require. Thus we require that

$$\begin{aligned} 0 &= \frac{dE}{dt} \\ &= \frac{du}{dt}\left(m\frac{d^2u}{dt^2} + V'(u)\right). \end{aligned}$$

In order for E to be conserved we either must have no motion ($\frac{du}{dt} = 0$), which is boring, or we must have

$$m\frac{d^2u}{dt^2} + V'(u) = 0.$$

Thus any formula for $V(u)$ gives rise to a differential equation for the motion. It is clear that the differential equation (1.1) arises from the potential function $V(u) = \frac{1}{2}ku^2$. In general, the force associated to some potential function is $F = -V'(u)$.

Let's summarize the logic here. We define two types of energy: kinetic energy, given by the formula $\frac{1}{2}m\left(\frac{du}{dt}\right)^2$, and potential energy, which in the case of the

[1]We won't discuss here what energy *is*–to paraphrase Richard Feynmann (see his *Lectures on Physics*): We don't really know what it is; we just know that we can compute this number. . . and the number is always the same!

oscillator is $V = \frac{1}{2}ku^2$. We interpret the kinetic energy K as the "energy associated with being moving" and the potential energy with the "energy associated with being at a position other than $u = 0$. The total energy is $E = K + V$ and could, in principle, depend on time. We require, however, that the total energy E is constant in time. From this requirement we deduce that either nothing happens (that is, $\frac{du}{dt} = 0$) or that u satisfies (1.1).

Notice, by the way, that k has some funny units. If we measure displacement u in meters, time t in seconds, and mass m in kilograms, then k must have units of $(\text{kilograms})/(\text{seconds})^2$.

I now want to apply some sort of logic to a much more complicated problem: the oscillation of a string. The first thing we need to do is figure out how to describe the oscillation of a string using a function. We can accomplish this by supposing that the string is strung between two points with distance $2L$ between them. We'll put these points on the x axis, one at $x = -L$ and one at $x = L$. If the string is "at rest" then it simply lies on the x axis. However, if the string is moving, then parts of it will be displaced from the x axis. We assume that this displacement happens in a plane, and let u measure the displacement. Since the displacement depends both on location x and on the time t, we have $u = u(t, x)$.

We now want to define some sort of "energy" for the string. First, we focus on kinetic energy. Let's assume that the string has a linear mass density that is constant in x, even as the string moves. (Think about how reasonable this is. . . it's actually better than it might first appear!) Call the linear mass density ρ. The little piece of string at location x has mass $\rho\, dx$ and vertical velocity $\frac{\partial u}{\partial t}$. It makes sense that the "kinetic energy" at that point would be $\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 dx$ and that the total kinetic energy of the string would be

$$K = \frac{1}{2}\rho \int_{-L}^{L} \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

Note that ρ is the "linear-densitized" version of the constant m appearing in the simple harmonic oscillator.

Let's now think about "potential energy," which is supposed to measure the "energy" associated with the location of the string. That is, we just saw a picture of a string at some time, we should be able to compute the potential energy without knowing anything about the velocity at that time. Another way to think about potential energy is this: Suppose that the string is being held in some physical configuration.

To what extent would the fact that the string is in that configuration lead to physical motion (and, perhaps, kinetic energy) if it was suddenly released?

If the string started out lying on the x axis, we would not expect any motion to suddenly emerge, so we would like that configuration to have zero potential energy. Likewise, if the string–endpoints and all–was just shifted up by some fixed amount, then we again would not expect any motion. Thus we want to assign zero potential energy to sections of the string which are described by u being constant in x.

On the other hand, if initially there is a part of the string is bent steeply, then we expect there to be quite a bit of motion when the string is released. This motivates us to define the potential energy to be

$$V = \frac{1}{2}k \int_{-L}^{L} \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Notice that in order for K and V to have the same units, it must be that k has units of $(\text{mass})/(\text{time})^2(\text{length})$, which is a linear-densitized version of the corresponding constant for the simple harmonic oscillator.

The total energy $E = K + V$ is thus

$$E = \rho\frac{1}{2} \int_{-L}^{L} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + c^2 \left(\frac{\partial u}{\partial x}\right)^2 \right\} dx,$$

where the constant $c = \sqrt{k/\rho}$ has units of (length)/(time); below we see how to interpret c as a velocity of "oscillation propagation."

We now impose the condition that E be constant in time, which requires that

$$0 = \rho \int_{-L}^{L} \left\{ \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\right] \right\} dx. \tag{3.1}$$

Integrating by parts we see that

$$\int_{-L}^{L} \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}\right] dx = \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_{-L}^{L} - \int_{-L}^{L} \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} dx.$$

Since the endpoints of the string are fixed, we have that $\frac{\partial u}{\partial t} = 0$ at $x = -L$ and at $x = L$. Thus (3.1) becomes

$$0 = \rho \int_{-L}^{L} \frac{\partial u}{\partial t} \left\{ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \right\} dx.$$

Since we do not want to restrict the velocity $\frac{\partial u}{\partial t}$, we conclude that in order to have energy conserved we must have

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{3.2}$$

Equation (3.2) is called the ***(one-dimensional) wave equation***, and describes the amplitude of an oscillating string. The wave equation (3.2) is, in some important senses, analogous to the simple harmonic oscillator equation (1.1).

IMPORTANT POINT 3.1. *It is useful to compare the simple harmonic oscillator and the wave equation:*

- *The oscillation of a single particle can be described by* (1.1)*; the oscillation of a string can be described by* (3.2)*.*
- *One can obtain the simple harmonic oscillator equation from conservation of energy; one can also deduce the wave equation from conservation of energy.*

There are many phenomena which might be described as "wave-like." This makes it difficult to formulate a precise definition of "waves" and "wave equations." One way to characterize wave equations is to require that "disturbances or signals propagate at some finite speed." The exercises below explore this idea[2].

Exercises

Exercise 3.1. Suppose $\phi_1 \colon \mathbb{R} \to R$ is a twice-differentiable function. Show that $u_1(t, x) = \phi_1(x - ct)$ is a solution to (3.2). Explain how one can interpret the solution as a "spatial shape begin manipulated in time."

Exercise 3.2. As an example, take $\phi_1(z) = e^{-z^2}$ and let $u_1(t, x) = \phi_1(x - ct)$ be the corresponding solution to (3.2). Make plots of $u_1(0, x), u_1(1, x), u_1(2, x), u_1(3, x)$. At what speed and direction does the shape travel? (For additional fun, choose a value of c and have a computer animate the solution!)

Exercise 3.3. Suppose that ϕ_2 is a second twice-differentiable function. Show that $u_2(t, x) = \phi_2(x + ct)$ is a solution to (3.2).

As an example, consider $\phi_2(z) = 2e^{-4(z-10)^2}$. Make a plot of the corresponding solution $u_2(t, x)$. At what speed and direction does the shape travel?

[2]This discussion of what makes a wave equation is motived by Roger Knobel's excellent book *An Introduction to the Mathematical Theory of Waves*, published by the American Mathematical Society.

Exercise 3.4. Show that (3.2) is linear (as defined in Exercise 1.3). Use this to conclude that $u_1(t,x) + u_2(t,x)$ is a solution to (3.2).

Exercise 3.5. Make a detailed plot (or, better yet, an animation) of the solution $u_1(t,x) + u_2(t,x)$. Describe the behavior; it might be useful to compare to the individual behavior of the solutions u_1 and u_2.

One can "physically" interpret the linearity property as "individual waves do not interact." Sometimes this is called the *superposition principle*. Do "real world waves" that behave this way? Discuss several different types of examples (water waves, sound waves, light, etc.).

Exercise 3.6. In the exercises above, all of the solutions travel at the same speed (namely, at speed c). Here we show that by using linearity we can obtain some very interesting solutions.

For simplicity, we set $c = 1$. Consider the solution

$$u(t,x) = \cos(x - t) + \alpha \cos(x + t),$$

where α is some parameter.

(1) First, show that when $\alpha = 0$ the solution is simply a traveling cosine wave. Use the following Mathematica code to animate the solution:

```
\[Alpha] = 0;
Animate[Plot[{Cos[(x-t)] + \[Alpha]*Cos[(x+t)]},
 {x,-8,8}, PlotRange->2], {t,0,50}]
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(2) Now consider the case when $\alpha = 0.1$. Explain how to interpret the the solution as the sum of two traveling waves, and explain the visual effect of the second wave.
(3) Repeat for $\alpha = 0.2, 0.3, 0.4, \ldots, 0.9$. What happens as you slowly increase α?
(4) Now set $\alpha = 1$. What is the behavior of the solution?
(5) Use the trigonometric identity for the sum of cosines to show that when $\alpha = 1$ the solution can be written

$$u(t,x) = 2\cos(t)\cos(x). \tag{3.3}$$

Explain how to interpret this as a "spatial shape that is changing in time."
(6) Solutions such as (3.3) are called *standing waves*. Explain why this name is appropriate. Then explain how to reconcile the idea of a "standing wave" with the idea that for wave equations "disturbances and signals propagate at some finite speed." [Note: We have set the speed c equal to 1 in this exercise; thus stating that the speed is zero is not an option!]

CHAPTER 4

Standing waves and the Fourier series hypothesis

In the previous chapter, we saw that an oscillating string can be described by a function $u(t, x)$ satisfying the one-dimensional wave equation (3.2). In this chapter we study this wave equation in more detail.

For simplicity, we set the constant c equal to 1. Thus we are interested in function $u(t, x)$ that is defined for

$$t \geq 0 \quad \text{and} \; -L \leq x \leq L, \tag{4.1}$$

and that satisfies the *partial differential equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{4.2}$$

as well as the *boundary conditions*

$$u(t, -L) = 0 \quad \text{and} \quad u(t, L) = 0. \tag{4.3}$$

4.1. Separation of variables

Motivated by our experience with the simple harmonic oscillator, we begin our investigation by looking for *standing wave solutions*, a concept you encountered in Exercise 3.6. Standing wave solutions are a type of *scaling solution* (something we saw earlier for the simple harmonic oscillator) and take the form

$$u(t, x) = A(t)\psi(x). \tag{4.4}$$

We call the function $A(t)$ the *amplitude function* of the scaling solution.

Inserting (4.4) in to (4.2) we find that

$$\underbrace{\frac{1}{A}\frac{d^2 A}{dt^2}}_{\clubsuit} = \underbrace{\frac{1}{\psi}\frac{d^2 \psi}{dx^2}}_{\heartsuit}.$$

We now make an important observation: the quantity ♣ is a function of t only, while the quantity ♡ is a function of x only. Thus

$$\frac{d}{dt}[\clubsuit] = \frac{d}{dt}[\heartsuit] = 0,$$

which means that ♣ is a constant and, since ♣ = ♡, that ♡ is equal to that same constant. Just as in the case of the simple harmonic oscillator, we call the constant λ.

Through the above analysis we have learned the following: *If* there is a solution to (4.2) of the form (4.4), *then* there must be some constant λ such that

$$\frac{d^2 A}{dt^2} = \lambda A \quad \text{and} \quad \frac{d^2\psi}{dx^2} = \lambda\psi. \tag{4.5}$$

At this point we are very excited – both of the equations in (4.5) are *linear ordinary differential equations*, which we know how to solve using the knowledge we gained in our differential equations course!

However, upon further inspection of the situation, we notice that we do not know what value λ should take. This is a problem – the solutions to (4.5) are very different depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$; see Exercise 4.1 below. We also notice that in our excitement about standing waves, we have neglected the condition (4.3). Although it is not obvious at first glance, it turns out that these two issues are related.

We remark that this procedure of looking for solutions to partial differential equations that are products of functions of one variable is called *separation of variables*. The advantage of this method is that it reduces partial differential equations to ordinary differential equations.

4.2. Boundary conditions and eigenvalue problems

We now address the issues of boundary conditions and the value of λ. We begin by first inserting our standing wave assumption (4.4) in to the condition (4.3). The result is that we require

$$A(t)\psi(-L) = 0 \quad \text{and} \quad A(t)\psi(L) = 0$$

for all times $t \geq 0$. The only way to both satisfy this condition and allow $A(t)$ to change in time is to require

$$\psi(-L) = 0 \quad \text{and} \quad \psi(L) = 0. \tag{4.6}$$

This means that we want ψ to satisfy

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= \lambda\psi \quad \text{for } -L \leq x \leq L, \\ \psi(-L) &= 0, \quad \text{and} \quad \psi(L) = 0. \end{aligned} \tag{4.7}$$

The problem of finding a function ψ and a number λ satisfying (4.7) is called a *Dirichlet eigenvalue problem*. Let's carefully explain this name.

Dirichlet: The condition that the function ψ is zero at the boundary of the interval $[-L, L]$ is called the ***Dirichlet boundary condition***. Physically, this condition corresponds to the fact that the endpoints of our string are fixed. Other boundary conditions are discussed in the exercises below.

Eigenvalue problem: The differential equation for ψ can be abstractly described as taking the form

$$[\text{operation}](\text{object}) = (\text{number})(\text{object}). \tag{4.8}$$

Given some operation, the problem of finding an object and number satisfying (4.8) is called an ***eigenvalue problem***.

Values of "number" for which (4.8) has a *non-zero* solution "object" are called ***eigenvalues***. Corresponding objects that solve (4.8) are called ***eigen-objects***.

It is important to notice parallels between the case of the simple harmonic oscillator and the wave equation: For the simple harmonic oscillator, the search for scaling solutions lead to an eigenvalue problem where the "object" was a 2D vector and the "operation" was multiplication by the matrix M. In the case of the wave equation, the "operation" refers to the second derivative, "object" refers to ψ. The parallels between these two problems is not a coincidence – rather, it is an example of an more general phenomenon.

IMPORTANT POINT 4.1 (SCALING SOLUTIONS AND EIGENVALUE PROBLEMS). *Scaling solutions to linear equations take the form $A(t)(object)$, where $(object)$ satisfies an eigenvalue problem. Thus the search for scaling solutions reduces to the search for solutions to the eigenvalue problem.*

In view of Important Point 4.1, we now focus our attention on (4.7). While the differential equation for ψ has solutions for every value of λ, the Dirichlet boundary condition at $x = \pm L$ restricts the possible values of λ. To see this we follow a two-step process. First we show that λ must be negative. Then we find a countable list of eigenvalues λ and eigenfunctions ψ solving (4.7).

To see that λ must be negative, we use an integration by parts argument. Assume that λ and ψ satisfy (4.7). We compute

$$\begin{aligned}\int_{-L}^{L} \psi(x)\lambda\psi(x)dx &= \int_{-L}^{L} \psi(x)\frac{d^2\psi}{dx^2}(x)\,dx \\ &= \left[\psi(x)\frac{d\psi}{dx}(x)\right]_{-L}^{L} - \int_{-L}^{L}\left(\frac{d\psi}{dx}(x)\right)^2 dx.\end{aligned}$$

Using the boundary condition, this implies that

$$\lambda \underbrace{\int_{-L}^{L} (\psi(x))^2\,dx} = -\underbrace{\int_{-L}^{L}\left(\frac{d\psi}{dx}(x)\right)^2 dx}\,. \tag{4.9}$$

Since both of the integrals are non-negative, it must be the case that $\lambda \le 0$.

The identity (4.9) also shows that if $\lambda = 0$ then $\frac{d\psi}{dx} = 0$, which implies that ψ is constant. However, the only constant function satisfying the Dirichlet boundary condition is the zero function. Thus we must have $\lambda < 0$.

Since $\lambda < 0$ we can write $\lambda = -\omega^2$ for some positive constant ω. Thus ψ must satisfy

$$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0.$$

From our differential equations course we know that solutions to this equation are

$$\cos(\omega x) \quad \text{and} \quad \sin(\omega x).$$

In order for $\cos(\omega x)$ to satisfy the Dirichlet boundary conditions we must have $\cos(\omega L) = 0$, which means that ω is an odd multiple of $\pi/2L$. Similarly, in order for $\sin(\omega x)$ to satisfy the boundary condition we must have ω be an even multiple of $\pi/2L$. Thus (4.7) has a nonzero solution ψ precisely when λ is one of

$$\lambda_k = -\omega_k^2 = -\left(\frac{\pi k}{2L}\right)^2, \quad k = 1, 2, 3, \ldots. \tag{4.10}$$

The corresponding solutions to (4.7) are listed in Figure 4.1.

We emphasize the following interplay between boundary conditions and the eigenvalue problem; see also the exercises below.

IMPORTANT POINT 4.2. *Imposing boundary conditions on an eigenvalue problem has two effects. First, an argument that combines integration by parts and the boundary conditions can be used to show that the eigenvalue must have a sign. Second, the boundary condition implies that there is only a discrete list of possible eigenvalues.*

TABLE 4.1. Dirichlet eigenvalues and eigenfunctions satisfying (4.7).

Eigenvalue λ	Eigenfunction ψ	counter
$\lambda_{2l+1} = -\left(\frac{\pi(2l+1)}{2L}\right)^2$	$\psi_{2l+1}(x) = \cos\left(\frac{\pi(2l+1)}{2L}\,x\right)$	$l = 0, 1, 2, 3\ldots$
$\lambda_{2l} = -\left(\frac{\pi l}{L}\right)^2$	$\psi_{2l}(x) = \sin\left(\frac{\pi l}{L}\,x\right)$	$l = 1, 2, 3, \ldots$

4.3. Standing wave solutions

For each of the solutions in Figure 4.1 equation (4.5) gives us a corresponding differential equation for the amplitude function $A(t)$. Since $\lambda_k = -\omega_k^2$, it is easy to see that the amplitude function $A_k(t)$ corresponding to spatial function $\psi_k(x)$ must be a linear combination of $\cos(\omega_k t)$ and $\sin(\omega_k t)$. Thus we obtain a long list of standing wave (or scaling) solutions to (4.2); see Figure 4.2.

This is quite an accomplishment – we have successfully found all of the standing wave solutions to (4.2)–(4.3). We now highlight a number of features of this process, and result, that exemplify important general phenomenon in the mathematics of oscillations.

First, we observe that the numbers ω_k tell us the frequency with which the standing waves oscillate. These numbers are sometimes called the *fundamental frequencies* or *natural frequencies* of the string. Remember that these numbers ω_k come from the eigenvalues of the operation $\frac{d^2}{dx^2}$; see (4.7).

IMPORTANT POINT 4.3 (EIGENVALUES TELL US FUNDAMENTAL FREQUENCIES). *Suppose that we have an eigenvalue problem describing the spatial shape of a standing wave. Then the eigenvalues corresponding to solutions of the problem tell us the frequency at which those solutions oscillate.*

TABLE 4.2. Standing wave solutions to (4.2) satisfying the Dirichlet boundary condition (4.3).

Standing wave solution	counter
$\cos(\omega_{2l+1}t)\psi_{2l+1}(x) = \cos\left(\frac{\pi(2l+1)}{2L}\,t\right)\cos\left(\frac{\pi(2l+1)}{2L}\,x\right)$	$l = 0, 1, 2, 3\dots$
$\sin(\omega_{2l+1}t)\psi_{2l+1}(x) = \sin\left(\frac{\pi(2l+1)}{2L}\,t\right)\cos\left(\frac{\pi(2l+1)}{2L}\,x\right)$	$l = 0, 1, 2, 3\dots$
$\cos(\omega_{2l}t)\psi_{2l}(x) = \cos\left(\frac{\pi l}{L}\,t\right)\sin\left(\frac{\pi l}{L}\,x\right)$	$l = 1, 2, 3, \dots$
$\sin(\omega_{2l}t)\psi_{2l}(x) = \sin\left(\frac{\pi l}{L}\,t\right)\sin\left(\frac{\pi l}{L}\,x\right)$	$l = 1, 2, 3, \dots$

Second, notice that there is a *list* of eigenvalues for which (4.7) has a solution, and that the list of eigenvalues in (4.10) are such that $\lambda_k \to -\infty$ as $k \to \infty$. Physically, this means that there is no upper bound on the frequency with which a standing wave can oscillate. Notice also that those standing waves which oscillate rapidly (i.e. those corresponding to large eigenvalues) have very high spatial variation.

IMPORTANT POINT 4.4. *For a finite oscillating object, we expect there to be a countable infinite, discrete, unbounded list of eigenvalues corresponding to standing waves. We furthermore expect higher frequencies to correspond to finer spatial variations.*

Finally, we note that the superposition (or linearity) principle applies to the wave equation (4.2); see Exercise 3.4. In particular, if $u_1(t, x)$ and $u_2(t, x)$ are solutions, and if α_1 and α_2 are constants, then

$$\alpha_1 u_1(t, x) + \alpha_1 u_2(t, x)$$

is also a solution.

Physically, the superposition principle means that the presence of one wave does not affect the behavior of another wave. You can observe this for yourself by creating two disturbances in a pan of water and watching the two "pass through" one another. Mathematically, it means that we can build new solutions by rescaling solutions and by adding together previously existing solutions. For example

$$u(t,x) = 17\cos\left(\frac{\pi}{L}t\right)\sin\left(\frac{\pi}{L}x\right) + \frac{1}{\sqrt{5}}\cos\left(\frac{3\pi}{L}t\right)\sin\left(\frac{3\pi}{L}x\right)$$

is a solution to the wave equation that is built from two standing waves. Notice that this solution is **not** a standing wave solution!

IMPORTANT POINT 4.5. *The linearity of the wave equation implies that we may build a large number of solutions to the wave equation by scaling and adding* finite combinations *of standing wave solutions.*

4.4. The initial value problem and the Fourier series hypothesis

By taking combinations of the standing wave solutions we can construct large numbers of solutions to the wave equation. In the case of the simple harmonic oscillator, we can in fact build all solutions by taking combinations of scaling solutions. This raises the question: Can every solution to the wave equation be constructed from standing wave solutions?

To investigate this question, we take a closer look at those solutions that can be constructed from standing waves. Since these solutions are finite combinations of standing wave solutions, we can write a generic combination as

$$u_N(t,x) = \sum_{k=1}^{N}\Big(\alpha_k\cos(\omega_k t)\psi_k(x) + \beta_k\sin(\omega_k t)\psi_k(x)\Big) \tag{4.11}$$

for some constants α_k and β_k.

When studying ordinary differential equations, we determined the constants appearing in "general solutions" using initial conditions. For a second-order ordinary differential equation, the initial value problem specified the initial position and velocity. This motivates us to try a similar approach for the wave equation.

The appropriate version of the initial value problem for the vibrating string is the Dirichlet ***initial boundary value problem (IBVP)***, which seeks to find a function u satisfying the following

$$\begin{aligned}
\text{wave equation:}\quad & \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\\
\text{Dirichlet boundary condition:}\quad & u(t,-L) = 0,\quad u(t,L) = 0,\\
\text{initial conditions:}\quad & u(0,x) = u_0(x),\quad \frac{\partial u}{\partial t}(0,x) = v_0(x),
\end{aligned}$$

where the functions $u_0(x)$ and $v_0(x)$ are specified and represent the initial shape and velocity of the wave.

By construction, the function $u_N(t, x)$ given by (4.11) already satisfies the wave equation and the Dirichlet boundary condition. We also compute that the shape of the solution at $t = 0$ is given by

$$u_N(0, x) = \sum_{k=1}^{N} \alpha_k \psi_k(x),$$

while the velocity at $t = 0$ is given by

$$\frac{\partial u_N}{\partial t}(0, x) = \sum_{k=1}^{N} \beta_k \omega_k \psi_k(x).$$

Thus the initial conditions in the IBVP are satisfied only if

$$\begin{aligned} u_0(x) &= \sum_{k=1}^{N} \alpha_k \psi_k(x) \\ v_0(x) &= \sum_{k=1}^{N} \beta_k \omega_k \psi_k(x). \end{aligned} \tag{4.12}$$

Hence we see that initial conditions determine the constants in (4.11) if and only if there is a unique way to express the functions u_0 and v_0 as the sum of a finite number of the functions ψ_k.

Important Point 4.6. *The ability to solve the initial boundary value problem using standing wave solutions is equivalent to being able to construct an arbitrary function u_0 out of the functions ψ_k.*

Unfortunately, there are many functions $u_0(x)$ that cannot be expressed as a finite sum of the functions $\psi_k(x)$. A good example is the function $u_0(x) = L - |x|$, which represents an initial shape of a string that has been pinched in the middle and pulled up.

At first, this may seem to derail our plan to construct solutions to the initial boundary value problem using sums of standing wave solutions. However, while the function $u_0(x) = L - |x|$ cannot be *exactly* expressed as the finite sum of the functions $\psi_k(x)$, it can be *approximated* by such a sum. It is easy to verify that by choosing

$$\alpha_k = \begin{cases} 2\left(\frac{2L}{k\pi}\right)^2 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

then the function

$$\sum_{k=1}^{N} \alpha_k \psi_k(x)$$

very closely approximates $u_0(x)$; see Figure 4.1. (Notice that this sum only involves cosines. For example, we have

$$\sum_{k=1}^{7} \alpha_k \psi_k(x) = 2\left(\frac{2L}{\pi}\right)^2 \cos\left(\frac{\pi x}{2L}\right) + 2\left(\frac{2L}{3\pi}\right)^2 \cos\left(\frac{3\pi x}{2L}\right) + 2\left(\frac{2L}{5\pi}\right)^2 \cos\left(\frac{5\pi x}{2L}\right) + 2\left(\frac{2L}{7\pi}\right)^2 \cos\left(\frac{7\pi x}{2L}\right).$$

This is because $u_0(x)$ is an even function, and $\psi_k(x)$ is even only when k is odd.)

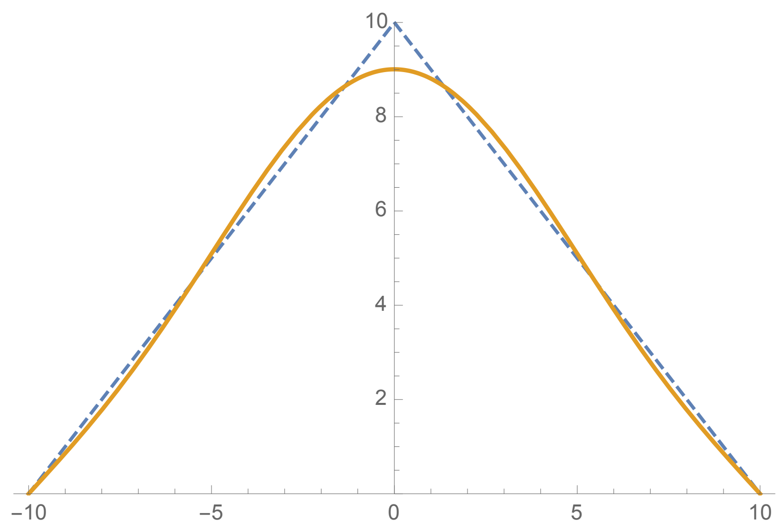

FIGURE 4.1. Plots of the function $u_0(x) = L - |x|$ (dashed) and the sum $\sum_{k=0}^{7} \alpha_k \psi_k(x)$ (solid) with $L = 10$. Even though the sum consists of only four cosine functions, it is a reasonable approximation of the function $u_0(x)$.

In 1822, Joseph Fourier published his work *Théorie analytique de la chaleur*, in which he claimed that any function can be approximated by a suitable sum of trigonometric functions. . . and that if infinite sums are allowed, then the sum is in fact equal to the function. In other words, he proposed constructing functions from an infinite series of trigonometric functions, much in the same way that many functions can be constructed from a power series consisting of an infinite sum of polynomials. I call this claim the “Fourier series hypothesis.”

Fourier's claim is not quite true in the way that he stated it in 1822, in large part due to the fact that many ideas in calculus were not well formulated at the time. (In fact, as you learn in the real analysis course, Fourier's claim inspired the development of much of the modern theory that puts calculus on a more solid foundation.) However, his hypothesis is "essentially true" and is one of the most important ideas in applied mathematics today.

In the following part of this course, we investigate in much more detail the sense in which the Fourier series hypothesis is true. For now, let us just observe that one consequence of the Fourier series hypothesis is that we can use standing waves to solve the initial boundary value problem for the wave equation.

Important Point 4.7. *If any function can be expressed as an infinite sum of the trigonometric functions $\psi_k(x)$, then we may solve the initial boundary value problem for the wave equation by carefully choosing the constants α_k, β_k in* (4.11) *and then taking the limit as $N \to \infty$.*

In other words, the study of generic oscillations can be accomplished by studying the standing wave solutions!

Exercises

Exercise 4.1. In our analysis of standing waves we showed that the Dirichlet boundary condition implies that the constant λ appearing in (4.5) is negative. Here we investigate the implications of λ being negative by comparing to the situation where λ is not negative.

(1) Suppose $\lambda = 0$. Describe the behavior of solutions to $\frac{d^2A}{dt^2} = \lambda A$.
(2) Suppose $\lambda > 0$. Describe the behavior of solutions to $\frac{d^2A}{dt^2} = \lambda A$.
(3) Are either of the cases above consistent with oscillatory behavior? What is the connection between the sign of the eigenvalue λ and oscillatory behavior?

Exercise 4.2. The Dirichlet boundary condition (4.6) is only one of several interesting boundary conditions. In this exercise, you explore the ***periodic boundary condition***:

$$\psi(-L) = \psi(L) \quad \text{and} \quad \frac{d\psi}{dx}(-L) = \frac{d\psi}{dx}(L).$$

A function satisfying the periodic boundary condition can be extended periodically to the whole real line. Thus waves satisfying the periodic boundary condition can be viewed as representing homogeneous waves on a very large domain.

(1) Show that the function $f(x) = x^3 - x$ satisfies the periodic boundary condition on the interval $[-1, 1]$. Then extend the graph of f to all of $\mathbb{R}$ by periodic repeating. Plot the periodic extension of $f'(x)$ on this extended domain as well.

(2) Find all standing wave solutions $u(t, x)$ to the wave equation (4.2) that satisfy the periodic boundary condition

$$u(t, -L) = u(t, L) \quad \text{and} \quad \frac{\partial u}{\partial x}(t, -L) = \frac{\partial u}{\partial x}(t, L). \tag{4.13}$$

Do this by completing the following steps:

- Show that the boundary condition (4.13) implies that ψ must satisfy the periodic boundary condition.
- Use integration by parts, and the periodic boundary condition, to show that the periodic eigenvalue problem only has solutions when $\lambda \leq 0$.
- Show that $\lambda_0 = 0$ is a possible eigenvalue, and that the corresponding solution to the eigenvalue problem is $\psi_0(x) = 1$.
- Show that the remaining eigenvalues are $\lambda_k = -\left(\frac{\pi k}{L}\right)^2$, $k = 1, 2, 3, \ldots,$ and that for each eigenvalue λ_k there are two independent eigenfunctions:
$$v_k(x) = \cos\left(\frac{\pi k}{L}x\right) \quad \text{and} \quad w_k(x) = \sin\left(\frac{\pi k}{L}x\right).$$
- Show that the standing waves satisfying (4.2) and (4.13) are
$$\begin{gathered} \psi_0, \\ \cos(\omega_k t) v_k(x), \\ \sin(\omega_k t) v_k(x), \\ \cos(\omega_k t) w_k(x), \\ \sin(\omega_k t) w_k(x), \end{gathered} \tag{4.14}$$
where $\omega_k = \frac{\pi k}{L}$. With what frequency does the first wave oscillate?
- Explain how the Fourier series hypothesis implies that a typical solution (4.2) with periodic boundary conditions can be written
$$\alpha_0 + \sum_{k=1}^{\infty} \beta_k \cos(\omega_k t) v_k(x) + \sum_{k=1}^{\infty} \gamma_k \sin(\omega_k t) v_k(x) + \sum_{k=1}^{\infty} \delta_k \cos(\omega_k t) w_k(x) + \sum_{k=1}^{\infty} \epsilon_k \sin(\omega_k t) w_k(x)$$

for some constants $\alpha_0, \beta_k, \gamma_k, \delta_k, \epsilon_k$.

(3) Standing waves with periodic boundary conditions have a feature that was not present with Dirichlet boundary conditions – each eigenvalue λ_k has two corresponding eigenfunctions $v_k(x) = \cos(\omega_k x)$ and $w_k(x) = \sin(\omega_k x)$. These function arise as solutions to the differential equation

$$\frac{d^2\psi}{dx^2} = -\omega_k^2 \psi.$$

From the differential equations course we know that we can also express solutions to this equation using complex numbers. Thus if we are willing to work with complex numbers the two eigenfunctions for λ_k are

$$e^{i\frac{\pi k}{L}x} \quad \text{and} \quad e^{-i\frac{\pi k}{L}x}.$$

Thus the functions

$$\psi_k(x) = e^{i\frac{\pi k}{L}x}, \quad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

are complex eigenfunctions such that λ_k corresponds to both ψ_k and ψ_{-k}.

- Explain how Euler's identity gives us a relationship between v_k, w_k and ψ_k, ψ_{-k}.
- Show that the complex "standing wave solutions" are given by

$$e^{i\frac{\pi k}{L}(x-t)} \quad \text{and} \quad e^{i\frac{\pi k}{L}(x-t)}. \tag{4.15}$$

- Notice that the real (or imaginary) parts of the complex solutions (4.15) are *not* actually standing waves. What combination of the complex solutions is needed to construct each of the standing waves listed in (4.14)?
- Look back at Exercise 3.6 and make some "intelligent remarks."

Exercise 4.3. Another important boundary condition is the ***Neumann boundary condition***, which in the case of the one-dimensional wave equation is

$$\frac{\partial u}{\partial x}(t, -L) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t, L) = 0.$$

One may physically interpret the Neumann boundary condition as a "perfect reflection" condition. Find all standing wave solutions to (4.2) satisfying the Neumann boundary condition.