Chapter 17

The Fourier transform

17.1 Definition

- We motivate the definition of the Fourier transform by considering the large L limit of the Fourier series

- First, we recall the Riemann sum definition of the integral:

$$\int_{-\infty}^{\infty} g(\xi)\, d\xi = \lim_{\Delta\xi \to 0} \sum_{k=-\infty}^{\infty} G(k\Delta\xi)\, \Delta\xi.$$

- For a function u in $L^2([-L, L])$ we have (for most x)

$$\begin{aligned} u(x) &= \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{2L} \int_{-L}^{L} u(y) e^{-i\frac{k\pi}{L}y}\, dy \right\} e^{i\frac{k\pi}{L}x} \\ &= \sum_{k=-\infty}^{\infty} \left[\left(\frac{1}{2\pi} \int_{-L}^{L} u(y) e^{-i\frac{k\pi}{L}y}\, dy \right) e^{i\frac{k\pi}{L}x} \right] \frac{\pi}{L}. \end{aligned}$$

  With $\Delta\xi = \frac{\pi}{L}$ we can view this as the Riemann sum of the integral

$$\int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} u(y) e^{-i\xi y}\, dy \right] d\xi$$

- Let's take a closer look at the term inside the round brackets. First consider

$$\frac{1}{2\pi}\int_{-L}^{L} u(y)e^{-i\frac{k\pi}{L}y}\,dy = \frac{f(u)_k}{\Delta\xi}.$$

  Since $f(u)_k$ represents the extent to which the function u is constructed from frequency $k\frac{\pi}{L} = k\Delta\xi$, we can understand this integral as the amount of frequency $k\Delta\xi$ in u per unit frequency.

- Taking the limit $L \to \infty$ we have

$$\frac{1}{2\pi}\int_{-L}^{L} u(y)e^{-i\frac{k\pi}{L}y}\,dy \to \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)e^{-i\xi y}\,dy$$

  which represents the "normalized" extent to which the function u is constructed from frequency ξ.

- This motivates the definition of the Fourier transform taking function u to the function $\mathcal{F}[u]$ defined by

$$\mathcal{F}[u](\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)e^{-i\xi y}\,dy.$$

  The Fourier transform is defined for function u satisfying

$$\int_{-\infty}^{\infty} |u(x)|\,dx < \infty \tag{17.1}$$ L1-finite

  We interpret $\mathcal{F}[u](\xi)$ to be the extent to which u has frequency ξ.

- Let $\mathcal{L}^2(\mathbb{R})$ be the functions in $L^2(\mathbb{R})$ satisfying (17.1). Then we can view $\mathcal{F}$ as a linear transformation $\mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$.

- Notation: It is common to denote $\mathcal{F}[u]$ by $\hat{u}$.

- Example: Triangle wave.

Exercise 17.1.

1. *Find a constant C so that the function*

$$u(x) = \begin{cases} C & \text{if } |x| \leq a \\ 0 & \text{otherwise} \end{cases}$$

   has norm $\|u\| = 1$.

2. *Compute the Fourier transform of u.*

3. *What happens to* $\hat{u}$ *as* $a \to \infty$*? What happens as* $a \to 0$*?*

Exercise 17.2. *Compute the Fourier transform of the following functions.*

1. $u(x) = e^{-|x|}$

2. $u(x) = e^{-ax^2}$*, where a is some positive constant. [Hint: Complete the square.]*

17.2 Properties of Fourier transform

The Fourier transform has properties similar to those of periodic Fourier series. . . and also some additional properties that are possible due to the infinite domain / lack of need for periodicity.

- Scaling property. Fix a function $u(x)$ and let $v(x) = u(ax)$. Then

$$\hat{v}(\xi) = \frac{1}{a}\hat{u}(\xi/a)$$

  We can interpret this as follows: Concentration in physical space leads to spread in frequency space and vice versa.

Exercise 17.3. *Translation property. Fix a function* $u(x)$ *and let* $w(x) = u(x - a)$*. Find a formula that relates* $\hat{u}$ *and* $\hat{w}$*. What does spatial translation do in frequency space?*

We also have properties for multiplication and derivation that are similar to those for the periodic Fourier series.

- Multiplication. Define

$$(u * v)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(y)v(x-y)\,dy.$$

  We have $\mathcal{F}[u * v] = \mathcal{F}[u] \cdot \mathcal{F}[v]$

Exercise 17.4. *1. Show that*

$$\mathcal{F}[u'](\xi) = i\xi\mathcal{F}[u](\xi).$$

2. Use the previous result to show that the Fourier transform turns the one dimensional wave equation in to an ODE for each frequency. The function u satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

if an only if

$$\frac{d^2 \hat{u}}{dt^2} = -\xi^2 \hat{u}.$$

What is the general solution to this ODE?

Our goal is to use the transform of the wave equation in order to understand waves on the real line. In order to do this we need to understand how to invert the Fourier transform.