Chapter 16

Fourier series revisited

16.1 Fourier series as a linear transformation

- Define vector space $l^2(\mathbb{Z})$ to be sequences $\alpha = \{\alpha_k\}$ of complex numbers such that

$$\|\alpha\|^2 = \sum_{k=-\infty}^{\infty} |\alpha_k|^2 < \infty$$

This is an inner product space with

$$\langle \alpha, \beta \rangle = \sum_{k=-\infty}^{\infty} \alpha_k \overline{\beta_k}.$$

- Example: $\alpha_k = \frac{(i)^k}{k!}$

- We view Fourier series as a transformation from $L^2([-L, L]) \to l^2(\mathbb{Z})$. Call this transformation the "little Fourier transform" and give it the letter f.

- Remember: For a function u in $L^2([-L, L])$ we extend periodically to $\mathbb{R}$.

- Notation: If u is a function in $L^2([-L, L])$ then $f(u)$ is a sequence of complex numbers. We write $f(u)_k$ for the k^{th} number.

- We have
$$f(u)_k = \frac{1}{2L}\int_{-L}^{L} u(x)e^{-i\frac{k\pi}{L}x}\,dx.$$
- f is a linear transformation
- f is an isomorphism: $\|f(u)\| = \frac{1}{\sqrt{2L}}\|u\|$.
- Inversion. f^{-1} takes sequences to functions by
$$f^{-1}(\alpha)(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{i\frac{k\pi}{L}x}.$$
- The convergence property from Chapter ?? implies that
$$f^{-1}(f(u))(x) = u(x)$$
at all points x where u is continuous, etc.
- Example: For some constant a with $0 < a < L$ define
$$u(x) = \begin{cases} \frac{1}{\sqrt{2a}} & \text{if } |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$
Note that $\|u\| = 1$.

  We compute
$$f(u)_k = \begin{cases} \frac{\sqrt{2a}}{2L} & \text{if } k = 0, \\ \frac{?}{k?}\sin\left(\frac{k\pi}{L}x\right) & \text{if } k \neq 0. \end{cases}$$
The isomorphism property means that
$$\sum_{k=-\infty}^{\infty} |f(u)_k| = \frac{1}{2L}$$
- size of k corresponds to spatial/frequency scale
- Return to earlier example: Notice what the shape of u is when a is small /

large.

We can plot $f(u)_k$ for various values of k...

- What happens if we have just a few small k values? What happens if we only have the large k values? High-pass / low-pass filters.

Exercise 16.1. *Set $L = 1$ and consider function*

$$u(x) = \begin{cases} (\#)(a - |x|) & \text{if } |x| < a, \\ 0 & \text{otherwise}. \end{cases}$$

1. *Find the number* $(\#)$ *so that* $\|u\| = 1$.
2. *Compute* $f(u)$
3. *Make a plot of* $f(u)_k$ *for large/small values of* a.
4. *Now set* $a = 1/2$. *Make a plot of the partial sum*

   $$\sum_{|k| \leq 5} f(u)_k e^{ik\pi x}.$$

   How closely does this describe u*?*
5. *Still with* $a = 1/2$. *Make a plot of the partial sum*

   $$\sum_{5 \leq |k| \leq 100} f(u)_k e^{ik\pi x}.$$

   What parts of the function u *does this describe?*

16.2 Properties of little Fourier transform

Three properties: products, even/odd, derivatives

- Products: Direct computation (using periodic extension)

$$\begin{aligned} f(u)_k f(v)_k &= \frac{1}{(2L)^2}\int_{-L}^{L}\left[\int_{-L}^{L} u(x)v(y)e^{-i\frac{k\pi}{L}(x+y)}\,dx\right]dy \\ &= \frac{1}{2L}\int_{-L}^{L}\left[\frac{1}{2L}\int_{-L}^{L} u(z)v(x-z)\,dz\right]e^{i\frac{k\pi}{L}x}\,dx \end{aligned}$$

- Convolution product: A new way to multiply functions

$$(u * v)(x) = \frac{1}{2L}\int_{-L}^{L} u(z)v(x-z)\,dz.$$

  Homework: Show that $u * v = v * u$.

- Little Fourier transform takes convolution product of functions to pointwise product of sequences:

$$f(u * v)_k = f(u)_k f(v)_k$$

- Example: What is convolution of square wave and triangle wave?

- Interpret convolution as “maximally dispersed multiplication” This is compatible with our earlier heuristic. . .

- (Optional / HW) If $u(x)$ is even then . . . we can use cosines. If $u(x)$ is odd then . . . we can use sines.

- Derivatives (require u to satisfy periodic BC):

$$f(u')_k = i\frac{k\pi}{L} f(u)_k.$$

16.3 Applications of properties

- Example: Suppose we want to solve

$$\frac{d^2u}{dx^2} = -\omega^2 u$$

with periodic BC.

Apply f to both sides in order to get

$$-\left(\frac{k\pi}{L}\right)^2 f(u)_k = -\omega^2 f(u)_k.$$

Thus we only have a solution when $\omega = \frac{k\pi}{L}$ for some k. All the other terms in $f(u)$ must be zero.

- Example/Homework: Suppose we want to solve

$$\frac{d^2u}{dx^2} = -\omega^2 u + h$$

with periodic BC. Here h is some forcing function.

1. Apply little Fourier transform to see that solution u must satisfy

$$f(u)_k = \frac{1}{\omega^2 - \left(\frac{k\pi}{L}\right)^2} f(h)_k.$$

2. Conclude that we do indeed have a periodic solution. . . unless there is a resonance!
3. Suppose that h is the square wave (and that there is no resonance). What is the solution?
4. Repeat for triangle wave forcing.