# Chapter 13

# Applications

## 13.1 Boundary value problems for the Dirichlet energy

Recall the optimization problem (11.4), which seeks to find a minimizer $u$ for the Dirichlet energy functional

$$E[u] = \frac{1}{2}\int_{\Omega} \|\operatorname{grad} u\|^2 \, dV$$

subject to the boundary condition

$$u|_{\Gamma} = b, \qquad (13.1)$$

AbstractDirichlet-BC

where $\Gamma$ is the boundary of $\Omega$ and $b\colon \Gamma \to \mathbb{R}$ is some specified function.

We address this problem in three steps. First we work invariantly to compute the gradient of the Dirichlet energy functional, arriving at a differential equation that any critical function $u$ must satisfy. Second, we show that there exists at most one solution to the differential equation satisfying the boundary condition (13.1). This implies that any critical point we can find is indeed the minimizer. Finally, in Chapter 14 we consider specific examples of domains $\Omega$ and boundary conditions for which we are able to construct solutions using Fourier-type series.

We now compute the gradient of the Dirichlet energy functional. Let $u_\epsilon = u + \epsilon\eta$ be a compact variation of function $u$. Using the product rule, we compute

$$\frac{d}{d\epsilon}E[u_\epsilon] = \frac{d}{d\epsilon}\frac{1}{2}\int_\Omega (\operatorname{grad} u_\epsilon)\cdot(\operatorname{grad} u_\epsilon)\,dV = \int_\Omega \operatorname{grad} u_\epsilon \cdot \operatorname{grad}\eta\,dV.$$

In order to proceed, we recall the following product rule from vector calculus: If $f$ is a function and $\mathbf{v}$ is a vector, then

$$\operatorname{Div}(f\mathbf{v}) = \operatorname{grad} f\cdot\mathbf{v} + f\operatorname{Div}\mathbf{v}. \tag{13.2}$$

divergence-product-rul

Using this identity with $\mathbf{v} = \operatorname{grad} u$ and $f = \eta$, we find

$$\frac{d}{d\epsilon}\left[E[u_\epsilon]\right]_{\epsilon=0} = \int_\Omega \left\{\operatorname{Div}(\eta\operatorname{grad} u) - \eta\operatorname{Div}(\operatorname{grad} u)\right\}\,dV.$$

Recall that $\eta$ vanishes in a neighborhood of the boundary $\Gamma$ of $\Omega$. Thus from the divergence theorem we have

$$\int_\Omega \operatorname{Div}(\eta\operatorname{grad} u)\,dV = \int_\Gamma \eta\operatorname{grad} u\cdot\hat{n}\,dA = 0,$$

where $\hat{n}$ is the unit normal to $\Gamma$. Consequently, we find that

$$\frac{d}{d\epsilon}\left[E[u_\epsilon]\right]_{\epsilon=0} = \int_\Omega -\eta\operatorname{Div}(\operatorname{grad} u)\,dV = \langle -\operatorname{Div}(\operatorname{grad} u), \eta\rangle$$

and thus that the gradient of the Dirichlet energy is

$$\operatorname{grad} E[u] = -\operatorname{Div}(\operatorname{grad} u). \tag{13.3}$$

AbstractDirichlet-grad

The operator $\operatorname{Div}(\operatorname{grad}(\cdot))$ is very important in both mathematics and physics, and is called the ***Laplace operator*** or "Laplacian." It is easy to compute in Cartesian coordinates that

$$\operatorname{Div}(\operatorname{grad} u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots.$$

**Warning 13.1.** *You need to be warned that there are different sign conventions and* *symbols used for the Laplacian.*

MyLaplacian

1. *In this course, we use the symbol $\Delta$ for the Laplacian as defined above. Thus in Cartesian coordinates, we have*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots .$$

*This convention is one of two common notations used by mathematicians. In particular, Paul uses this convention in his research.*

LizLaplacian

2. *It is also common for mathematicians to define the Laplacian to have the opposite sign. This alternate convention is motivated by the negative sign that shows up in* (13.3) *(and the desire for positive eigenvalues). Thus in research by many people (including some of Paul's collaborators) the Cartesian coordinate expression is*

$$-\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots .$$

*There are also some mathematicians who define the divergence to have the opposite sign. . . , but we won't get in to that issue here.*

3. *Finally, in physics it is common to use the symbol $\nabla^2$ for the Laplacian. Thus in many physics texts you will find the Cartesian expression*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots .$$

*(Paul has to confess that it hurts him a little bit to type that.)*

Now that we have compute the gradient of the Dirichlet energy functional, the problem of finding minimizers of $E[u]$ satisfying (13.1) is reduced to the following boundary value problem for a partial differential equation:

Laplace-DirichletBVP

$$\Delta u = 0 \tag{13.4a}$$

$$u|_{\Gamma} = b. \tag{13.4b}$$

The first equation in (13.4) is called ***Laplace's equation***.

We now show that solutions to (13.4) are unique. To do this, suppose that both $u_1$ and $u_2$ satisfy (13.4). The function $w = u_1 - u_2$ satisfies

$$\Delta w = 0 \quad \text{and} \quad w|_\Gamma = 0.$$

Using (13.2) we see that

$$\operatorname{Div}(w \operatorname{grad} w) = \operatorname{grad} w \cdot \operatorname{grad} w + w\Delta w = \|\operatorname{grad} w\|^2.$$

Integrating this over the region $\Omega$ yields

$$\int_\Omega \operatorname{Div}(w \operatorname{grad} w)\, dV = \int_\Omega \|\operatorname{grad} w\|^2\, dV.$$

Using the divergence theorem, and the fact that $w = 0$ along the boundary $\Gamma$, we see that

$$\int_\Omega \|\operatorname{grad} w\|^2\, dV = \int_\Gamma w \operatorname{grad} w \cdot \hat{n}\, dA = 0.$$

Since $\|\operatorname{grad} w\|^2 \geq 0$ we conclude that in fact we must have $\operatorname{grad} w = \mathbf{0}$. This, in turn, means that $w$ is a constant function. Since $w = 0$ at the boundary, it must mean that $w = 0$, which means that $u_1 = u_2$.

While the coordinate-invariant definition of the Laplacian is useful for showing properties of solutions to Laplace's equation, it is also convenient to have expressions in coordinates. It is rather straightforward to compute $\Delta u$ in Cartesian coordinates directly from the definition, but for polar and spherical coordinates it is often easier to derive an expression for $\Delta u$ using the calculus of variations.

**Example 13.2.** *Suppose $\Omega$ is a region in $\mathbb{R}^2$ and that we are making use of polar coordinates $r$ and $\theta$. We can compute the Laplacian $\Delta u$ as follows.*

*The expression for the Dirichlet energy is*

$$E[u] = \frac{1}{2}\iint_*^* \left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right\} r\, dr\, d\theta.$$

*If $u_\epsilon = u + \epsilon\eta$ is a variation of $u$, then we have*

$$\frac{d}{d\epsilon}E[u_\epsilon] = \iint_*^* \left\{ \left(\frac{\partial u_\epsilon}{\partial r}\right)\left(\frac{\partial \eta}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial u_\epsilon}{\partial \theta}\right)\left(\frac{\partial \eta}{\partial \theta}\right) \right\} r\, dr\, d\theta.$$

*We now proceed by evaluating at $\epsilon = 0$ and using integration-by-parts. The first term in the integrand we integrate by parts with respect to $r$, while the second term we integrate by parts with respect to $\theta$. The result is*

$$\begin{aligned} \frac{d}{d\epsilon}\left[E[u_\epsilon]\right]_{\epsilon=0} &= \iint_*^* \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right\} \eta\, r\, dr\, d\theta \\ &= \left\langle \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \eta \right\rangle. \end{aligned}$$

*Thus in polar coordinates we have*

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

*which we can also write as*

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{13.5}$$

polar-Laplacian

*We conclude this example by observing that, in two dimensions, we have*

$$\Delta(\ln r) = 0.$$

**Exercise 13.3.** *Use the calculus of variations to compute $\Delta u$ in spherical coordinates. Show that in three dimensions we have $\Delta(r^{-1}) = 0$.*

## 13.2 Euler-Lagrange equations in classical mechanics

We now apply the calculus of variations to the action integral (11.6). Letting $\mathbf{u}_\epsilon = \mathbf{u} + \epsilon\eta$ be a variation of $\mathbf{u}$ we compute

$$\begin{aligned}\frac{d}{d\epsilon}A[\mathbf{u}_\epsilon] &= \int_0^T \left\{ m\frac{d\mathbf{u}_\epsilon}{dt}\cdot\frac{d\eta}{dt} - \operatorname{grad} V(\mathbf{u}_\epsilon)\cdot\eta \right\} dt \\ &= \int_0^T \left\{ -m\frac{d^2\mathbf{u}_\epsilon}{dt^2} - \operatorname{grad} V(\mathbf{u}_\epsilon) \right\}\cdot\eta\, dt,\end{aligned}$$

where we have integrated by parts in the second step.

Evaluating at $\epsilon = 0$ we see that $\mathbf{u}$ is a critical point of $A[\mathbf{u}]$ if

$$m\frac{d^2\mathbf{u}}{dt^2} = -\operatorname{grad} V(\mathbf{u}). \tag{13.6}$$ generic-EL

The equation (13.6) is called the ***Euler-Lagrange equation***, or equation of motion, for the action integral (11.6). If one defines the term $-\operatorname{grad} V(\mathbf{u})$ to be the force present on the particle, then (13.6) is simply Newton's second 'law' $m\mathbf{a} = \mathbf{F}$.

**Example 13.4.** *Consider, for example, the action integral* (11.7). *In this case* $V(u) = mgu$ *and thus* $-\operatorname{grad} V(u)$ *is simply* $-V'(u) = -mg$. *Thus the Euler-Lagrange equation is*

$$m\frac{d^2u}{dt^2} = -mg.$$

*This reduces to* $u''(t) = -g$. *Making use of the initial conditions, it is straightforward to see that the solution to* (11.8) *is*

$$u(t) = u_0 + v_0 t - \frac{1}{2}gt^2.$$

*Note that, per the discussion in Example* **??**, *the solution does not depend on the mass* $m$ *of the particle!*

**Example 13.5.** *Consider* (11.10), *which is a functional of the functions* $x$ *and* $y$. *We make use of the variations* $x_\epsilon = x + \epsilon\eta_x$ *and* $y_\epsilon = y + \epsilon\eta_y$ *and compute as* *follows:*

$$\frac{d}{d\epsilon}A[x_\epsilon, y_\epsilon] = \int_0^T \left\{ m\frac{dx_\epsilon}{dt}\frac{d\eta_x}{dt} + m\frac{dy_\epsilon}{dt}\frac{d\eta_y}{dt} - \frac{GMm}{(x_\epsilon^2 + y_\epsilon^2)^{3/2}}\left(x\eta_x + y\eta_y\right)\right\} dt.$$

*Evaluating at* $\epsilon = 0$ *and integrating the first two terms in the integrand by parts yields*

$$\begin{aligned}\frac{d}{d\epsilon}&\left[A[x_\epsilon, y_\epsilon]\right]_{\epsilon=0}\\ &= \int_0^T \left\{ \left(-m\frac{d^2x}{dt^2} - \frac{GMm\,x}{(x^2+y^2)^{3/2}}\right)\eta_x + \left(-m\frac{d^2y}{dt^2} - \frac{GMm\,y}{(x^2+y^2)^{3/2}}\right)\eta_y\right\} dt.\end{aligned}$$

*This we write as*

$$\frac{d}{d\epsilon}\left[A[x_\epsilon, y_\epsilon]\right]_{\epsilon=0} = \left\langle \begin{pmatrix} -m\frac{d^2x}{dt^2} - \frac{GMm\,x}{(x^2+y^2)^{3/2}} \\ -m\frac{d^2y}{dt^2} - \frac{GMm\,y}{(x^2+y^2)^{3/2}} \end{pmatrix}, \begin{pmatrix}\eta_x \\ \eta_y\end{pmatrix} \right\rangle,$$

*from which we conclude that the Euler-Lagrange equations are*

$$\begin{aligned} -m\frac{d^2x}{dt^2} - \frac{GMm\,x}{(x^2+y^2)^{3/2}} &= 0,\\ -m\frac{d^2y}{dt^2} - \frac{GMm\,y}{(x^2+y^2)^{3/2}} &= 0. \end{aligned}$$

**Exercise 13.6.** *Compute the Euler-Lagrange equations for* $r$ *and* $\theta$ *that arise from the action integral* (11.12)*. Compare to the assignment "Report: Gravitation" that was assigned in the differential equations course.*

## 13.3 The wave equation

We now show that the wave equation arises as the Euler-Lagrange equation of the action integral (11.13). Before making the generic computation, let's work out the Euler-Lagrange equation in the one-dimensional setting.

**Example 13.7.** *Suppose we are working in one dimension, with* $\Omega = (a, b)$*. In this* *setting,* (11.13) *is simply*

$$A[u] = \frac{1}{2}\int_0^T \int_a^b \left\{ \left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 \right\} dx\, dt.$$

*Letting* $u_\epsilon = u + \epsilon\eta$ *be a variation of* $u$ *we compute*

$$\frac{d}{d\epsilon} A[u_\epsilon] = \int_0^T \int_a^b \left\{ \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial \eta}{\partial t}\right) - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) \right\} dx\, dt.$$

*Integrating by parts we find*

$$\frac{d}{d\epsilon}\left[A[u_\epsilon]\right]_{\epsilon=0} = \int_0^T \int_a^b \left(-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2}\right) \eta\, dx\, dt$$

*and thus we see that the Euler-Lagrange equation is indeed*

$$-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0. \tag{13.7}$$

In order to compute the Euler-Lagrange equation for (11.13), we make use of the fact that the action integral can be written $A[u] = K[u] - E[u]$. We have already computed the gradient of the Dirichlet energy $E[u]$, thus it remains to compute the gradient of the total kinetic energy $K[u]$.

Let $u_\epsilon = u + \epsilon\eta$ be a variation of $u$. We compute

$$\begin{aligned}
\frac{d}{d\epsilon} K[u_\epsilon] &= \frac{d}{d\epsilon}\int_0^T \int_\Omega \frac{1}{2}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dV\, dt \\
&= \int_0^T \int_\Omega \frac{\partial u_\epsilon}{\partial t}\frac{\partial \eta}{\partial t}\, dV\, dt \\
&= -\int_0^T \int_\Omega \frac{\partial^2 u_\epsilon}{\partial t^2}\eta\, dV\, dt.
\end{aligned}$$

Using this, we conclude that

$$\frac{d}{d\epsilon}\left[A[u_\epsilon]\right]_{\epsilon=0} = \int_0^T \int_\Omega \left(-\frac{\partial^2 u}{\partial t^2} + \Delta u\right) \eta\, dV\, dt.$$

Thus the Euler-Lagrange equation for the action integral (11.13) is

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = 0. \tag{13.8}$$

generic-wave

Equation (13.8) is called the ***wave equation***.

It is useful to have expressions for (13.8) in various coordinate systems. In Cartesian coordinates, the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots. \tag{13.9}$$

Using (13.5) we see that in two dimensions, the polar coordinate expression for the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

**Exercise 13.8.** *Find the expression of the wave equation in three-dimensional spherical coordinates.*

## 13.4 Gradient flows and the heat equation

We conclude this chapter by presenting another use of gradients, called "gradient flows."

Let begin by supposing that we have a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. The gradient vector field $\operatorname{grad} f(\mathbf{x})$ tells us, at each point $\mathbf{x}$, the direction in which $f$ is increasing the most. The negative gradient $-\operatorname{grad} f(\mathbf{x})$ tells us the direction in which $f$ is decreasing the most. Thus one strategy for minimizing the function $f$ would be to start at an arbitrary point $\mathbf{x}_0$ and follow a path such that along the path the tangent vector is $-\operatorname{grad} f(\mathbf{x})$. Such a path is called the ***gradient flow of*** $f$ and is described by the differential equation

$$\frac{d\mathbf{x}}{dt} = -\operatorname{grad} f(\mathbf{x}).$$

**Example 13.9.** *Consider the function* $f(x, y) = (x^2 - 4)^2 + 4 * y^2$. *The gradient of*

$f$ *is* grad $f(x, y) = \langle 4x^3 - 16x, 8y \rangle$ *and thus the gradient flow is described by*

$$\begin{aligned} \frac{dx}{dt} &= -4x^3 + 16x, \\ \frac{dy}{dt} &= -8y. \end{aligned} \tag{13.10}$$

*The contour plot of* $f$*, together with a plot of the vector field* $-$ grad $f(x, y)$*, is given in Figure 13.1.*

*Notice that, in general, the gradient flow will find either the critical point* $(2, 0)$ *or the critical point* $(-2, 0)$*. However, if the initial point* $(x_0, y_0)$ *has* $x_0 = 0$*, then the gradient flow will tend towards* $(0, 0)$*.*

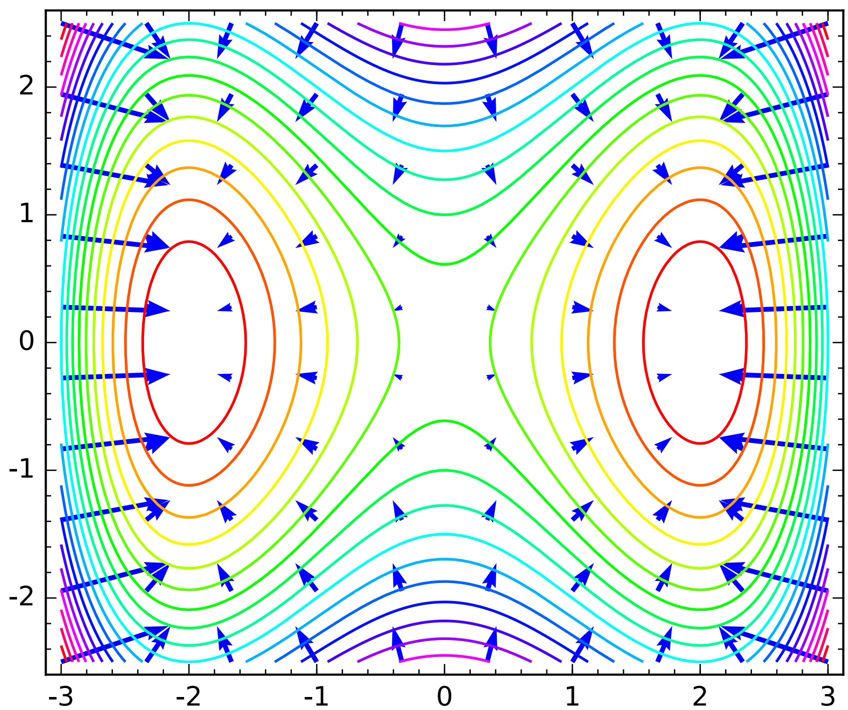


Figure 13.1: The contour plot of the function $f(x, y) = (x^2 - 4)^2 + 4 * y^2$, together with the vector field $-$ grad $f(x, y)$. Notice. . .

`gradient-graphic01`

**Example 13.10.** *Consider the function* $f(x, y) = (x^2 - 4)(x + 1) + 4 * y^2$*. The* *contour plot, together with the vector field* $-\operatorname{grad} f(x, y)$*, is shown in Figure 13.2.*

*Notice that if the initial condition* $(x_0, y_0)$ *is too far to the left, then the gradient flow "misses" the local minimum and "falls off the cliff".*

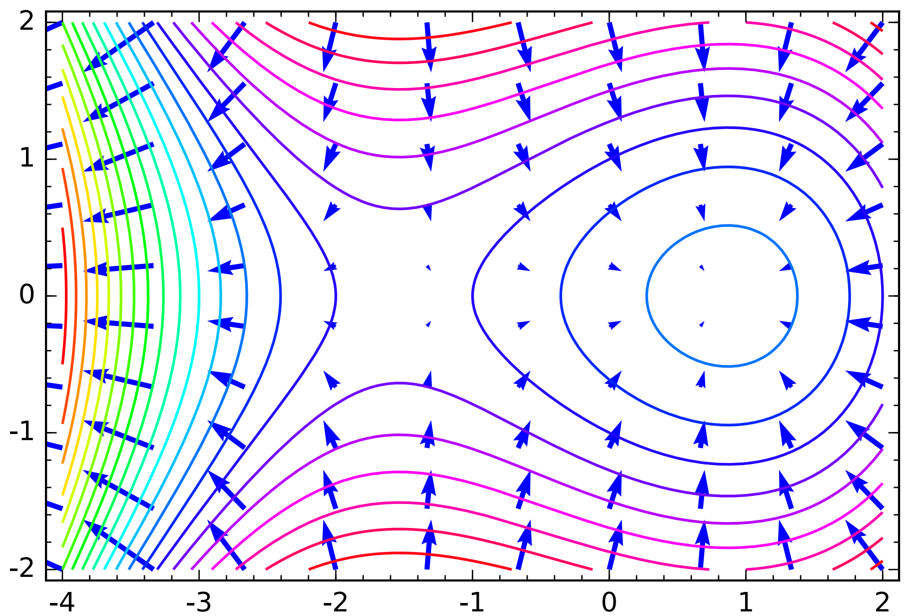


Figure 13.2: The contour plot of the function $f(x, y) = (x^2 - 4)(x + 1) + 4 * y^2$, together with the vector field $-\operatorname{grad} f(x, y)$. Notice. . . `gradient-graphic02`

Suppose now we have a functional $F$ that takes in functions $u\colon \Omega \to \mathbb{R}$. We define the $L^2$ ***gradient flow of*** $F$ to be the differential equation

$$\frac{\partial u}{\partial t} = -\operatorname{grad} F[u],$$

where $u\colon [0, T] \times \Omega \to \mathbb{R}$.

**Example 13.11.** *Consider the Dirichlet energy functional* $E[u]$ *acting on functions* $u$ *with domain* $\Omega$*. Since* $\operatorname{grad} F[u] = -\Delta u$*, the* $L^2$ *gradient flow of the Dirichlet energy is*

$$\frac{\partial u}{\partial t} = \Delta u. \tag{13.11}$$

`generic-heat`

*Equation* (13.11) *is called the* ***heat equation****.*

*In two-dimensional Cartesian coordinates, the heat equation is*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

*while in polar coordinates the heat equation is*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The behavior of solutions to gradient flow equations is rather different from the behavior of solutions to the wave equation. In particular, it is a common feature that gradient flow equations approach minimizers exponentially.

To see this, we first consider a one-dimensional example.

example:GF-basic **Example 13.12.** *Consider the function $f(x) = x^3 - 6x^2$. The corresponding gradient flow, $\frac{dx}{dt} = -f'(x)$, is the differential equation*

$$\frac{dx}{dt} = -3x^2 + 12x. \tag{13.12}$$ GF-basic

*Notice that there are two equilibrium solutions to* (13.12)*: $x = 0$ and $x = 4$. A stream plot for this equation appears in Figure 13.3.*

*If we linearize* (13.12) *about $x = 0$ we obtain the equation*

$$\frac{dy}{dt} = 12y$$

*which has solutions $y_0 e^{12t}$.*

*If we linearize about $x = 4$ we obtain the equation*

$$\frac{dy}{dt} = -12y,$$

*which has solution $y_0 e^{-12t}$.*

*Thus we see that solutions to the gradient flow equation* (13.12) *exhibit exponential* *growth near the unstable equilibrium $x = 0$, and decay exponentially towards the stable equilibrium $x = 4$. Thus, while not all solutions do find the minimum of $f$ at $x = 4$, those solutions that do find this equilibrium approach the minimizer exponentially. This is rather different than the oscillatory behavior we have observed previously.*

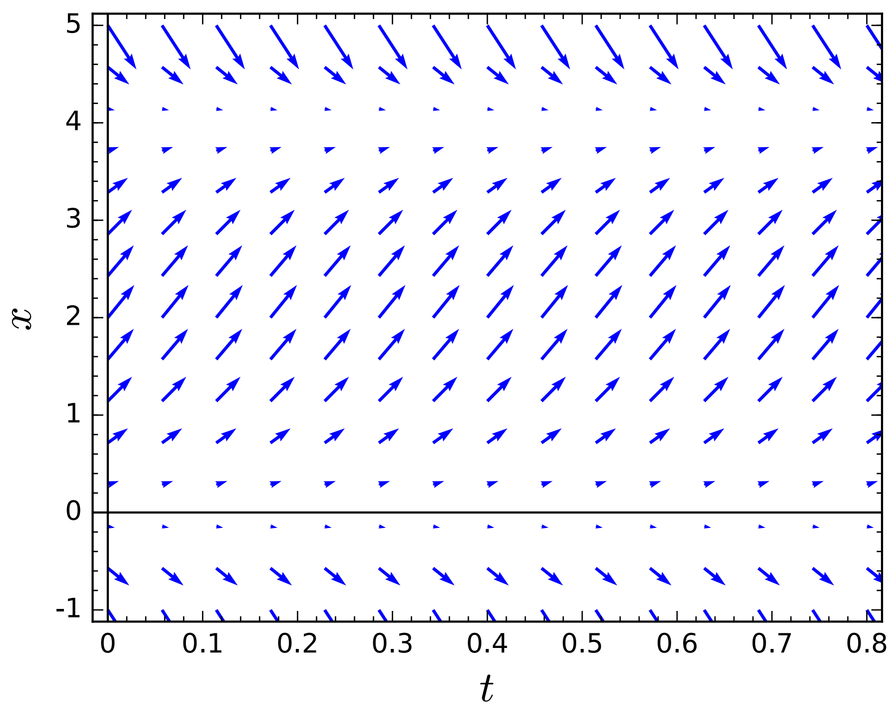


Figure 13.3: A stream plot for the gradient flow (13.12) showing solutions moving towards the local minimum $x = 4$. `GF-basic-01`

We now see that the exponential behavior observed in Example 13.12 also occurs for solutions to the heat equation. For simplicity, let us consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{13.13}$$

`1D-heat`

where $x$ is in the domain $\Omega = [-1, 1]$ and where we impose Dirichlet boundary conditions

$$u(t,-1) = 0 \quad \text{and} \quad u(t,1) = 0.$$

We start by looking for scaling solutions of the form $u(t,x) = A(t)\psi(x)$. Plugging this in to (13.13) we see that $A$ must satisfy

$$\frac{dA}{dt} = \lambda A$$

and $\psi$ must satisfy the Dirichlet eigenvalue problem

$$\frac{d^2\psi}{dx^2} = \lambda\psi, \qquad \psi(-1) = 0, \quad \psi(1) = 0.$$

Fortunately, we already know that the solutions to this eigenvalue problem occur when $\lambda$ is one of

$$\lambda_k = -\left(\frac{\pi k}{2}\right)^2$$

and

$$\psi_k(x) = \begin{cases} \cos\left(\frac{\pi k}{2}x\right) & \text{if } k \text{ is odd,} \\ \sin\left(\frac{\pi k}{2}x\right) & \text{if } k \text{ is even.} \end{cases}$$

Thus we see that the scaling solutions to (13.13) that satisfy the Dirichlet boundary conditions are

$$u(t,x) = e^{-\left(\frac{\pi(2l+1)}{2}\right)^2 t} \cos\left(\frac{\pi(2l+1)}{2}x\right) \quad \text{and} \quad u(t,x) = e^{-(\pi l)^2 t} \sin(\pi l x),$$

which each exponentially decay towards zero.