# Part III

# The calculus of variations

In this part of the course we develop tools that we can use to address the *optimization problem*. Before we begin, it is appropriate to ask: What is the optimization problem? What does this have to do with oscillations?

Suppose we have a collection of objects and a way to measure the "cost" (or, alternatively, "value") of each object. The optimization problem is to find those objects with the least cost (or, alternatively, greatest value).

In this class, we work with collections of objects that form a vector space. In this setting, a "way of measuring cost" means a function that assigns a number to each object in our vector space. The optimization problem is to find the object that corresponds to the smallest possible number.

Calculus provides some great tools for studying optimization. In first-year calculus class, we learn about optimization where the collection of objects is just the collection of real numbers, and thus the function that measures cost is simply a function $\mathbb{R} \to \mathbb{R}$. In order to optimize such a function, we look for critical points – inputs where the derivative of the function is zero.

In our multivariable calculus class, we learned how to define critical points of functions when the cos function has inputs in the vector space $\mathbb{R}^n$. One of the primary tasks of this part of the course is figuring out a way to define the concept of a critical point when the inputs are from a vector space of functions.

The other task of this part of the course is connecting the optimization problem to oscillations. At first glance, it may seem that the optimization and and oscillations have nothing to do with each other. As it turns out, the oscillations described by the wave equation are "as efficient as possible" and thus correspond to the solution to an optimization problem.

# Chapter 11

# Functions and functionals

## 11.1 Functions and optimization problems

We begin with two examples.

**Example 11.1.** *Suppose we want to describe the temperature at various locations in the plane. One way to do this would be to use a function* $f : \mathbb{R}^2 \to \mathbb{R}$*, assigning to each location* $\mathbf{x} \in \mathbb{R}^2$ *a temperature* $f(\mathbf{x})$*. The problem of optimizing the temperature now becomes the problem of finding the object* $\mathbf{x}$ *such that* $f(\mathbf{x})$ *is the largest/smallest.*

**Example 11.2.** *Suppose we have a map of some region and want to find the point with the highest elevation. We could do this by viewing our map as a subset of* $\mathbb{R}^2$ *and then constructing a function* $f(\mathbf{x})$ *that gives us the elevation at point* $\mathbf{x}$*. The problem then becomes finding the point* $\mathbf{x}$ *such that* $f(\mathbf{x})$ *is largest.*

In both Example 11.1 and Example 11.2 we phrase the optimization problem in the following form:

> Construct a function where the inputs come from some vector space and where the outputs are numbers. Then find the object in vector space that has the smallest/largest number.

The point of this chapter is to generalize this type of problem to the situation where the vector space is a vector space of functions (such as $L^2$). In a subsequent chapter we then turn to the task of actually finding the optimizer.

Functions where the inputs come from some vector space of functions and the output is a number are called ***functionals***.

In the next sections we consider three important types of functionals: geometric functionals, functionals related to classical mechanics, and functionals related to the wave equation.

For each functional we introduce, we also pose an associated optimization problem. Some of these optimization problems as that the functional be minimized; others ask that a certain function be a "critical point" of the functional in question. For the purposes of this chapter, we do not provide a precise definition of "critical point." A precise definition is presented in the following chapter.

## 11.2 Geometric functionals

We consider here functionals that describe geometric quantities.

**Example 11.3.** *Consider the collection of functions $u\colon [-1,1] \to \mathbb{R}$. We define a functional $L[u]$ that computes the length of the graph of the function $u$:*

$$L[u] = \int_{-1}^{1} \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx. \tag{11.1}$$

*A natural optimization problem associated to the functional* (11.1) *is the Dirichlet boundary value problem:*

> *Find the function $u$ that minimizes $L[u]$ subject to the Dirichlet boundary condition*
>
> $$u(-1) = a \quad \text{and} \quad u(1) = b,$$
>
> *where $a$ and $b$ are fixed numbers.*

The functional in Example 11.3 is related to the Dirichlet energy appearing in Chapter 4. Recall the Taylor expansion

$$\sqrt{1+\xi^2} \approx 1 + \frac{1}{2}\xi^2 + \dots .$$

Thus the linearization of the functional (11.1) is

$$2 + \frac{1}{2}\int_{-1}^{1}\left(\frac{du}{dx}\right)^2 dx.$$

Since the leading factor of 2 is independent of the function $u$, optimizing this last integral is equivalent to optimizing

$$E[u] = \frac{1}{2}\int_{-1}^{1}\left(\frac{du}{dx}\right)^2 dx. \tag{11.2}$$

The integral in (11.2) is called the ***one-dimensional Dirichlet energy*** and corresponds to the potential energy term appearing in the derivation of the wave equation in Chapter 4.

**Example 11.4.** *We can pose an optimization problem associated to* (11.2)*:*

> *Find the function $u$ that minimizes $E[u]$ subject to the Dirichlet boundary condition*
>
> $$u(-1) = a \quad \text{and} \quad u(1) = b,$$
>
> *where $a$ and $b$ are fixed numbers.*

We can generalize the previous example to curves in the plane that are not necessarily graphs over the $x$ axis.

**Example 11.5.** *Consider the collection of all paths $\mathbf{u}\colon [0,1] \to \mathbb{R}$. We define a functional $L[\mathbf{u}]$ that computes the length of the path:*

$$L[\mathbf{u}] = \int_0^1 \left\|\frac{d\mathbf{u}}{dt}\right\| dt.$$

*The following boundary value problem is an optimization problem associated with* *this functional:*

> *Fix points* $\mathbf{a} = (x_a, y_a)$ *and* $\mathbf{b} = (x_b, y_b)$ *in the plane. Find the function* $\mathbf{u}$ *such that* $\mathbf{u}(0) = \mathbf{a}$ *and* $\mathbf{u}(1) = \mathbf{b}$*, and that minimizes the length* $L[\mathbf{u}]$.

Examples 11.3, 1D-Dirichlet-dnergy-problem, and 11.5 can be generalized to higher-dimensional objects. For simplicity, we consider the following situation.

**Example 11.6.** *Let* $u \colon \Omega \to \mathbb{R}$*, where* $\Omega$ *is some region on* $\mathbb{R}^2$*. We construct the functional* $A$ *that measures the area of the graph of* $u$ *as follows.*

*The graph of* $u$ *is the surface parametrized by* $(x, y, u(x, y))$*. At each point, the coordinate tangent vectors are*

$$\partial_x = \langle 1, 0, \partial_x u\rangle \quad \text{and} \quad \partial_y = \langle 0, 1, \partial_y u\rangle.$$

*Thus the area element is*

$$dA = \begin{vmatrix} (\partial_x \cdot \partial_x) & (\partial_x \cdot \partial_y) \\ (\partial_y \cdot \partial_x) & (\partial_y \cdot \partial_y) \end{vmatrix}^{1/2} dx\,dy = \sqrt{1 + \|\operatorname{grad} u\|^2}\,dx\,dy$$

*and the area functional is*

$$A[u] = \int_\Omega \sqrt{1 + \|\operatorname{grad} u\|^2}\,dx\,dy$$

*Surfaces given by functions* $u$ *that are critical points of* $A[u]$ *are known as* ***minimal surfaces****.*

*The boundary value problem associated to the area functional is known as the* ***Plateau problem****:*

> *Let* $\Omega$ *be a region in* $\mathbb{R}^2$ *and let* $\Gamma$ *be the boundary of* $\Omega$*. Suppose* $b \colon \Gamma \to \mathbb{R}$*. Find the function* $u \colon \Omega \to \mathbb{R}$ *such that* $u|_\Gamma = b$ *and that minimizes the area* $A[u]$. (11.3)

**Example 11.7.** *As in the one-dimensional setting, we can linearize* $A[u]$ *about*

$u = 0$. *The result is the* ***two-dimensional Dirichlet energy functional***

$$E[u] = \frac{1}{2}\int_\Omega \|\operatorname{grad} u\|^2 \, dx \, dy.$$

*Functions that are critical points of the Dirichlet energy are called* ***harmonic functions***.

*We can pose a boundary value optimization problem for the Dirichlet energy as well:*

> *Let $\Omega$ be a region in $\mathbb{R}^2$ and let $\Gamma$ be the boundary of $\Omega$. Suppose $b\colon \Gamma \to \mathbb{R}$. Find the function $u\colon \Omega \to \mathbb{R}$ such that $u|_\Gamma = b$ and that minimizes the Dirichlet energy $E[u]$.* (11.4)

More generally, if $\Omega$ is a domain in $\mathbb{R}^n$ we can define the ***Dirichlet energy functional*** of a function $u\colon \Omega \to \mathbb{R}$ by

$$E[u] = \frac{1}{2}\int_\Omega \|\operatorname{grad} u\|^2 \, dV, \tag{11.5}$$

where $dV$ refers to the Euclidean volume element in $\Omega$.

It is helpful to have expressions for the Dirichlet energy is various coordinate systems. Various helpful formulas and conventions related to coordinate systems are listed in Appendix A.

**Example 11.8** (Dirichlet energy in Cartesian coordinates)**.** *Here we consider the two- and three-dimensional cases in Cartesian coordinates.*

1. *If $\Omega \subset \mathbb{R}^2$ then* (11.5) *takes the form*

$$E[u] = \frac{1}{2}\iint_*^* \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx \, dy$$

2. *If $\Omega \subset \mathbb{R}^3$ then* (11.5) *takes the form*

$$E[u] = \frac{1}{2}\iiint_*^* \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} dx \, dy \, dz$$

**Exercise 11.9** (Dirichlet energy in polar and spherical coordinates)**.**

1. *Suppose* $\Omega$ *is a region in* $\mathbb{R}^2$*. Find the expression for the Dirichlet energy in polar coordinates.*
2. *Suppose* $\Omega$ *is a region in* $\mathbb{R}^3$*. Find the expression for the Dirichlet energy in spherical coordinates.*

## 11.3 Functionals related to classical mechanics

Suppose a particle of mass $m$ is traveling along a path described by the function $\mathbf{u}\colon [0,T] \to \mathbb{R}^n$.

The ***total kinetic energy*** is

$$K[\mathbf{u}] = \int_0^T \frac{1}{2} m \left\| \frac{d\mathbf{u}}{dt} \right\|^2 \, dt.$$

If the potential energy is given by a potential function $V\colon \mathbb{R}^n \to \mathbb{R}$ then the ***total potential energy*** is

$$U[\mathbf{u}] = \int_0^T V(\mathbf{u}) \, dt.$$

Finally, the ***action integral*** $A[\mathbf{u}]$ is the difference between the total kinetic and potential energy:

$$A[\mathbf{u}] = K[\mathbf{u}] - U[\mathbf{u}] = \int_0^T \left\{ \frac{1}{2} m \left\| \frac{d\mathbf{u}}{dt} \right\|^2 - V(\mathbf{u}) \right\} dt. \tag{11.6}$$

One of the fundamental principles in classical mechanics is the "principle of least action," which roughly states that particles will travel in such a way to optimize the action integral $A[\mathbf{u}]$.

While we posed boundary value problems for the geometric functionals in the previous section, it is typical to pose initial value problems for functionals in classical mechanics:

> Suppose we are given a potential function $V(\mathbf{u})$, an initial location $\mathbf{u}_0$, and an initial velocity $\mathbf{v}_0$. Find a trajectory $\mathbf{u}(t)$ such that
>
> $$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \frac{d\mathbf{u}}{dt}(0) = \mathbf{v}_0,$$
>
> and that is a critical point of of the action integral $A[\mathbf{u}]$.

**Example 11.10.** *Consider a particle of mass $m$ that is moving vertically near the surface of the earth. Describe the vertical motion by the function $u\colon [0,T] \to \mathbb{R}$. Thus the total kinetic energy is*

$$K[u] = \int_0^T \frac{1}{2} m \left(\frac{du}{dt}\right)^2 \, dt.$$

*If we assume that the only phenomenon relevant to the particle's motion is the gravitation of the earth, then the total potential energy is*

$$U[u] = \int_0^T m\, g\, u \, dt.$$

*Hence the action integral is*

$$A[u] = \int_0^T \left\{ \frac{1}{2} m \left(\frac{du}{dt}\right)^2 - m\, g\, u \right\} dt. \tag{11.7}$$

*The optimization problem associated to this action integral is the following:*

> *Suppose we are given an initial height $u_0$ and an initial vertical velocity $v_0$. Find a function $u(t)$ such that*
>
> $$u(0) = u_0 \quad \textit{and} \quad u'(0) = v_0, \tag{11.8}$$
>
> *and such that $u$ is a critical point of the integral* (11.7).

*Notice that the mass $m$ of the particle factors out of the action integral. This means that we expect the optimal trajectory of the particle to be independent of $m$.*

**Example 11.11.** *Consider the example of a particle of mass $m$ traveling in the vicinity of another (stationary) object of mass $M$. The potential energy is given by*

$$V = \frac{GMm}{|\text{distance between two objects}|}.$$

*Thus if we assume that the stationary object is located at the origin, then*

$$U[\mathbf{u}] = -\frac{GMm}{\|\mathbf{u}\|}$$

*and the action integral is*

$$A[\mathbf{u}] = \int_0^T \left\{ \frac{1}{2} m \left\| \frac{d\mathbf{u}}{dt} \right\|^2 + \frac{GMm}{\|\mathbf{u}\|} \right\} dt. \tag{11.9}$$

*The corresponding optimization problem is the following:*

> *Given an initial location $\mathbf{u}_0$ and initial velocity $\mathbf{v}_0$ find a function $\mathbf{u}(t)$ such that*
>
> $$\mathbf{u}(0) = \mathbf{u}_0 \quad \textit{and} \quad \mathbf{u}'(0) = \mathbf{v}_0,$$
>
> *and that is a critical point of the action integral $A[\mathbf{u}]$.*

*Notice again that the mass $m$ of the traveling particle can be factored out of the integral... and thus the trajectory $\mathbf{u}(t)$ that optimizes $A$ does not depend on the mass of the particle.*

*Finally, suppose we restrict attention to the case that the particle travels in a plane. Using Cartesian coordinates, the trajectory $\mathbf{u}(t) = (x(t), y(t))$ and*

$$A[x, y] = \int_0^T \left\{ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 + \frac{GMm}{\sqrt{x^2 + y^2}} \right\} dt \tag{11.10}$$

*In this case the optimization problem becomes the following:*

> *Given initial location $(x_0, y_0)$ and initial velocity $(v_0, w_0)$, find functions $x(t)$ and $y(t)$ such that*
>
> $$x(0) = x_0, \quad y(0) = y_0, \quad x'(0) = v_0, \quad y'(0) = w_0, \tag{11.11}$$
>
> *and that are critical points of the functional* (11.10).

**Exercise 11.12.** *Suppose we restrict to motion in the plane and use polar coordinates, describing the path $\mathbf{u}(t)$ in terms of the functions $r(t)$ and $\theta(t)$. Show that the action integral* (11.9) *becomes*

$$A[r, \theta] = \int_0^T \left\{ \frac{1}{2} m \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right) + \frac{GMm}{r} \right\} dt. \tag{11.12}$$

We conclude this section with a comparison between Example 11.10 and Exercise 11.12. Suppose that the particle in Exercise 11.12 is traveling near the earth. Let $M_{\text{earth}}$ be the mass of the earth and let $r_{\text{earth}}$ be the radius of the earth. Linearizing the function $\frac{1}{r}$ at $r = r_{\text{earth}}$ yields

$$\frac{GM_{\text{earth}} m}{r} \approx \frac{GM_{\text{earth}} m}{r_{\text{earth}}} - \frac{GM_{\text{earth}} m}{r_{\text{earth}}^2} (r - r_{\text{earth}}).$$

Supposing that the motion of the particle is only in the radial direction and setting $u = r - r_{\text{earth}}$ we see that the functional (11.12) is approximated by a constant plus the integral

$$\int_0^T \left\{ \frac{1}{2} m \left( \frac{du}{dt} \right)^2 - m \left( \frac{GM_{\text{earth}}}{r_{\text{earth}}^2} \right) u \right\} dt.$$

This should be compared to the functional in Example 11.10. In particular, the reader is strongly encouraged to compare the constants

$$\frac{GM_{\text{earth}}}{r_{\text{earth}}^2} \quad \text{and} \quad g.$$

## 11.4 Action integral for the wave equation

We now introduce an action integral for the wave equation. Let $\Omega$ be some region in $\mathbb{R}^n$, and consider a function $u(t, \mathbf{x})$ defined for $t \in [0, T]$ and $\mathbf{x} \in \Omega$. Let $dV$ be the Euclidean volume element on $\Omega$.

We define the total kinetic energy of $u$ to be

$$K[u] = \frac{1}{2} \int_0^T \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 dV\, dt.$$

For the "potential" energy we make use of the Dirichlet energy functional, defining

$$U[u] = \int_0^T E[u]\, dT = \frac{1}{2} \int_0^T \int_\Omega \| \operatorname{grad} u \|^2\, dV\, dt.$$

It is important to note that the gradient of $u$ above is only with respect to the spatial variable $\mathbf{x}$. The action integral is defined to be

$$A[u] = K[u] - U[u] = \frac{1}{2} \int_0^T \int_\Omega \left\{ \left( \frac{\partial u}{\partial t} \right)^2 - \| \operatorname{grad} u \|^2 \right\} dV\, dt. \tag{11.13}$$

The optimization problems we pose for (11.13) have both the boundary-value features of the geometric functionals and the initial-value aspects of the classical mechanics problems. In particular, we pose the following problem:

> Suppose $\Omega$ is a bounded region with boundary $\Gamma$. Let $b \colon \Gamma \to \mathbb{R}$, $s \colon \Omega \to \mathbb{R}$, and $v \colon \Omega \to \mathbb{R}$ be specified functions. We seek a function $u(t, \mathbf{x})$ such that
>
> $$\begin{aligned} u(t, \mathbf{x}) &= b(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \Gamma \text{ and } t > 0, \\ u(0, \mathbf{x}) &= s(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \Omega \\ \frac{\partial u}{\partial t}(0, \mathbf{x}) &= v(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \Omega, \end{aligned}$$
>
> and such that $u$ is a critical point of the action integral (11.13).

It is helpful to have expressions for the action (11.13) in various coordinate systems.

**Example 11.13** (Wave action in Cartesian coordinates)**.** *Suppose* $\Omega$ *is the rectangular domain* $[-L, L] \times [-M, M]$ *in* $\mathbb{R}^2$*. Then the action integral* (11.13) *becomes*

$$A[u] = \frac{1}{2} \int_0^T \int_{-L}^{L} \int_{-M}^{M} \left\{ \left( \frac{\partial u}{\partial t} \right)^2 - \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial u}{\partial y} \right)^2 \right\} dt\, dx\, dy.$$

*The initial conditions consist of an initial shape* $s(x, y)$ *of the wave and an initial velocity* $v(x, y)$*.*

*The boundary conditions have four components:*

$$\begin{aligned} u(t, -L, y) &= b_1(y) \quad \text{for } -M < y < M, \\ u(t, x, -M) &= b_2(x) \quad \text{for } -L < x < L, \\ u(t, L, y) &= b_3(y) \quad \text{for } -M < y < M, \\ u(t, x, M) &= b_4(x) \quad \text{for } -L < x < L, \end{aligned}$$

*where* $b_1$*,* $b_2$*,* $b_3$*,* $b_4$ *are specified functions.*

**Exercise 11.14.**

1. *Suppose* $\Omega$ *is the unit disk in* $\mathbb{R}^2$*. Find the expression for the action integral* (11.13) *in polar coordinates. How should the initial and boundary conditions be specified in polar coordinates?*

2. *Suppose* $\Omega$ *is the unit ball in* $\mathbb{R}^3$*. Find the expression for the action integral* (11.13) *in spherical coordinates. How should the initial and boundary conditions be specified in spherical coordinates?*

## 11.5 Neumann boundary conditions (optional)