Chapter 9

Sturm-Liouville theory

ch:sturm-liouville

Sturm-Liouville theory is the mathematical framework describing situations that "behave like periodic Fourier series." To understand what this means, let's first recall what happened with periodic Fourier series.

9.1 Periodic Fourier series revisited

Our exploration of periodic Fourier series began with the periodic eigenvalue problem

$$\begin{aligned} \frac{d^2u}{dx^2} &= \lambda u \qquad \text{for} \qquad -1 < x < 1, \\ u(1) = u(-1) &\quad \text{and} \quad u'(1) = u'(-1). \end{aligned} \tag{9.1}$$

periodic-problem

Notice that we are only enforcing the differential equation on the interior of the domain.

The first thing we did was use an integration-by-parts argument to show that we must have $\lambda \leq 0$. Thus we set $\lambda = -\omega^2$ for some $\omega \geq 0$. A second integration-by-parts argument shows that solutions with different values of λ must be orthogonal with respect to the standard $L^2([-1, 1])$ inner product. Both of these integration-by-parts arguments make use of the boundary conditions.

We the went looking for solutions to this differential equations that had power series expansions. Putting a generic power series in to the differential equation, we found that solutions take the form

$$\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\omega x)^{2k} + \beta \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\omega x)^{2k+1},$$

where α and β are constants that can be freely chosen. We verified that these series converge absolutely for any value of x, thus justifying the use of power series.

We then proceeded by "observing" that these power series are, in fact, our old friends cosine and sine. Thus the typical solution to the ordinary differential equation is

$$\alpha \cos(\omega x) + \beta \sin(\omega x).$$

At this stage we imposed the periodic boundary condition, which implied that ω must be an integer multiple of π. Consequently, we obtained a sequence of eigenvalues $\lambda_k = -\omega_k^2 \leq 0$ such that $\lambda_k \to -\infty$ as $k \to \infty$, together with a sequence of orthogonal functions $v_k = \cos(k\pi x)$, $w_k = \sin(k\pi x)$.

With this collection of orthogonal functions in hand, we then considered the approximation problem: Given a function u, can we find constants α_k and β_k such that

$$\sum_k (\alpha_k \cos(k\pi x) + \beta_k \sin(k\pi))$$

converges to u? Using orthogonality, we found that the optimal choice of α_k and β_k and then set out to show convergence.

It was convenient in the proof of convergence to convert the problem to complex exponentials (though we know how to convert back & forth between complex exponentials and real cosines/sines). Using the Dirichlet kernel, we shows that we do indeed get pointwise convergence if u and u' are continuous; we also obtained a convergence result for the case that u and u' are only piecewise continuous. Finally, Paul said some words about convergence in norm. . . though we postponed those details.

In summary, the periodic Fourier series has the following pieces:

- an eigenvalue problem involving a second-order differential equation,
- boundary conditions that allow us to use integration by parts,
- existence of solutions using power series,
- a countably infinite list of eigenvalues, and
- a corresponding list orthogonal eigenfunctions that form a complete set.

The purpose of this chapter is to show that there are many eigenvalue problems that give rise to collections of complete orthogonal collections of functions. Such problems are called Sturm-Liouville problems.

9.2 The Sturm-Liouville theorem

The purpose of this section is to motivate the Sturm-Liouville theorem, which describes situations under which we have complete orthogonal collections of functions.

We work with the domain $\Omega = (a, b)$; typical examples include $\Omega = (-1, 1)$, $\Omega = (0, \infty)$, and $\Omega = (-\infty, \infty)$. On this domain, we use the weighted L^2 inner product

$$\langle u, v\rangle_w = \int_a^b u(x)v(x)w(x)\,dx, \tag{9.2}$$ SL-generic-ip

where w is a positive weight function. We denote the corresponding norm with a subscript, so that

$$\|u\|_w^2 = \langle u, u\rangle_w.$$

It is important to note that the domain Ω does not include the endpoints, and that we allow for the possibility that the weight function w vanishes at the endpoints.

On the domain Ω we consider the eigenvalue problem

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = \lambda u, \tag{9.3}$$ SL-generic-ode

where a, b, c are functions of x. The left side of (9.3) we write as Lu, where

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \tag{9.4}$$ SL-generic-L

is called a ***second-order linear operator***. Using this notation we write (9.3) as

$$Lu = \lambda u. \tag{9.5}$$ SL-generic

Example 9.1. *In the periodic eigenvalue problem* (9.1)*, the second order operator is* $L = \frac{d^2}{dx^2}$ *and the weight function is* $w(x) = 1$*. The domain is* $\Omega = (-1, 1)$.

The eigenvalue and eigenfunctions that solve the periodic eigenvalue problem (9.1) have two important properties:

1. eigenfunctions associated to different eigenvalues are orthogonal, and
2. all eigenvalues have the same sign.

We make two definitions related to these properties.

1. We say that a second order linear operator L is ***self-adjoint*** with respect to the inner product (9.2) if

   $$\langle Lu, v\rangle_w = \langle u, Lv\rangle_w$$

   for all test functions $u, v \in C_0^\infty(\Omega)$.

2. We say that a self-adjoint operator L is ***negative*** with respect to the inner product (9.2) if

   $$\langle Lu, u\rangle_w \le 0$$

   for all test functions $u \in C_0^\infty(\Omega)$.

Example 9.2. *The operator* $L = \frac{d^2}{dx^2}$ *is self-adjoint and negative with respect to the usual unweighted* L^2 *inner product on* $[-1, 1]$.

We proceed by determining what conditions we must put on the functions a, b, c in order to conclude that the operator L is self-adjoint and negative.

self-adjoint-condition **Exercise 9.3.** *Suppose that* $u, v \in C_0^\infty(\Omega)$. *(In particular, we are assuming that* u *and* v *are identically zero near the boundary of* Ω*.) Show, using integration by parts ♡, that*

$$\langle Lu, v\rangle_w = \langle u, Lv\rangle_w + \int_a^b u \left\{ 2\left(\frac{d(aw)}{dx} - bw\right)\frac{dv}{dx} + \frac{d}{dx}\left(\frac{d(aw)}{dx} - bw\right) v \right\}\, dx.$$

Conclude that L *is self-adjoint only if* $b = \dfrac{1}{w}\dfrac{d(aw)}{dx}$.

Using Exercise 9.3, we see that self-adjoint operators of the form (9.4) can be written in the form

$$Lu = \frac{1}{w}\left(\frac{d}{dx}\left[p\frac{du}{dx}\right] + ru\right), \tag{9.6}$$ SL-new-L

where p and r are functions related to a, b, and c by

$$a = \frac{p}{w} \qquad b = \frac{1}{w}\frac{dp}{dx} \qquad c = \frac{r}{w}.$$

The point of all this is that (9.4) has three freely-chosen functions (a, b, c). However, if one requires that the operator L be self-adjoint, then there are only two functions (p, r) that can be freely chosen.

Writing L in terms of the functions p and r as in (9.6), we can express (9.5) as

$$\frac{d}{dx}\left[p\frac{du}{dx}\right] + ru = \lambda w u. \tag{9.7}$$ SL-new-ode

Suppose now that $u \in C_0^\infty(\Omega)$. Using (9.6) and integrating by parts we see that

$$\langle Lu, u\rangle_w = \int_a^b \left\{ -p\left(\frac{du}{dx}\right)^2 + ru^2 \right\}\, dx.$$

Thus if we require

$$p(x) \geq 0 \quad \text{and} \quad r(x) \leq 0 \qquad \text{for } a < x < b, \tag{9.8}$$ $\boxed{\texttt{SL-negative-condition}}$

then the operator L is negative.

In view the the discussion above, we assume that L takes the form (9.6) for functions p and r satisfying (9.8). Consequently L is self-adjoint with respect to the inner product with weight w and the eigenvalue problem (9.5) takes the form (9.7).

We now turn to the issue of boundary conditions. The goal is to choose boundary conditions so that the self-adjoint and negative properties of L imply that the eigenfunctions of L are orthogonal and that the eigenvalues are all negative (or zero).

Exercise 9.4. *Suppose that L is a self-adjoint operator of the form* (9.6) *and that $Lu_1 = \lambda_1 u_1$, $Lu_2 = \lambda_2 u_2$. Show that* (9.7) *implies*

$$\lambda_1 \langle u_1, u_2 \rangle_w = p \frac{du_1}{dx} u_2 \bigg|_a^b - u_1 \frac{du_2}{dx} \bigg|_a^b + \lambda_2 \langle u_1, u_2 \rangle_w . \tag{9.9}$$ $\boxed{\texttt{SL-BC-ID}}$

If the boundary term in (9.9) vanishes, then we have

$$(\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle_w = 0$$

and thus $\lambda_1 \neq \lambda_2$ implies that u_1 and u_2 are orthogonal. This motivates the following definition. Given a self-adjoint operator L of the form (9.6), a boundary condition is called ***admissible*** if for any two functions u_1 and u_2 satisfying the boundary condition we have

$$\left[p \left(\frac{du_1}{dx} u_2 - u_1 \frac{du_2}{dx} \right) \right]_a^b = 0. \tag{9.10}$$ $\boxed{\texttt{SL-admissible-bc}}$

Example 9.5. *The periodic boundary condition on the domain $\Omega = (-1, 1)$ is admissible for the operator $L = \frac{d^2}{dx^2}$.*

In the case that the domain Ω is infinite, then we need to interpret the condition (9.10) as a condition on the rate of growth, measured in terms of the weight function.

Example 9.6. *Consider the case where* $\Omega = (-\infty, \infty)$ *and* $w(x) = e^{-x^2}$*. In order for*

$$\|u\|_w^2 = \int_{-\infty}^{\infty} u(x)^2 e^{-x^2}\, dx < \infty$$

we must have $\lim_{x\to\pm\infty} e^{-x/2}|u(x)| = 0$*. In this case, one way to satisfy* (9.10) *would be to impose the condition that* u *grow at some polynomial rate as* $x \to \pm\infty$*.*

In the following exercise, you show that if u is an eigenfunction of a negative, self-adjoint operator and if u satisfies an admissible boundary condition for that operator, then the corresponding eigenvalue cannot be positive.

Exercise 9.7. *Suppose that* L *is a negative, self-adjoint operator, and suppose that* u *satisfies* $Lu = \lambda u$ *and an admissible boundary condition for* L*. Show that*

$$\lambda \|u\|_w^2 \le 0$$

and thus that $\lambda \le 0$*.*

The following summarizes our discussion of operators and boundary conditions.

Theorem 9.8. *Let* $\Omega = (a, b)$ *and let* w *be a positive function on* w*. Suppose* L *is a self-adjoint, negative operator with respect to the inner product* (9.2)*. Then* L *takes the form* (9.6) *for functions* p *and* r*.*

Furthermore, if u_1 *and* u_2 *are eigenfunctions of* L *satisfying an admissible boundary condition, then*

1. *if the corresponding eigenvalues* λ_1*,* λ_2 *are different, then* $\langle u_1, u_2\rangle_w = 0$*; and*
2. *the corresponding eigenvalues satisfy* $\lambda_1, \lambda_2 \le 0$*.*

Finally, we state the Sturm-Liouville theorem, which tells us that, in fact, such orthogonal eigenfunctions exist, and that they form a complete set.

Theorem 9.9 (Sturm-Liouville theorem)**.** *Let* $\Omega = (a, b)$ *and let* w *be a positive weight function on* Ω*. Suppose that* L *is a self-adjoint, negative operator with* *respect to the inner product* (9.2). *Finally, let (BC) be an admissible boundary condition for L.*

Then there exists

1. *an infinite sequence of eigenvalues*

   $$0 \geq \lambda_1 > \lambda_2 > \lambda_3 > \dots$$

   such that $\lambda_k \to -\infty$ *as* $k \to \infty$ *and*

2. *a corresponding infinite sequence of orthogonal eigenfunction* ψ_k *such that* $L\psi_k = \lambda_k$ *such that the collection* $\{\psi_k\}$ *is complete in* $L^2(\Omega)$ *with respect to the norm arising from* (9.2).

In particular, the theorem tells us that if u is any function with $\|u\|_w < \infty$ then the series

$$\sum_{k=1}^{\infty} \alpha_k \psi_k, \qquad \text{where} \qquad \alpha_k = \frac{\langle u, \psi_k \rangle_w}{\|\psi_k\|_w^2}$$

converges in norm to u.

The proof of the Sturm-Liouville theorem is, unfortunately, beyond the scope of this course. Roughly speaking, the main points are the following:

- First, one needs to show existence of eigenfunctions using some abstract theory for differential equations.
- Second, one shows pointwise convergence of the series in the case that $u \in C_0^\infty(\Omega)$.
- Finally, one uses the fact that any function in $L^2(\Omega)$ can be approximated by a function in $C_0^\infty(\Omega)$.

If you are interested in some of the details, I recommend the book *Sturm-Liouville Theory and its Applications* by M.A. Al-Gwaiz, published in the Springer Undergraduate Mathematics Series.