Chapter 8

Convergence for periodic Fourier series

We are now in a position to address the Fourier series hypothesis – that functions can realized as the infinite sum of trigonometric functions – discussed at the end of 4.4. For simplicity, we work with domain $\Omega = [-1, 1]$ and use the periodic boundary condition. The theory for other boundary conditions is similar, if a small bit more complicated.

It is also convenient to work in the complex setting.

8.1 Periodic Fourier series

Recall from Example 6.25 the functions $\psi_k = e^{ik\pi x}$, which are orthogonal with respect to the standard inner product on $L^2([-1, 1])$ and satisfy the periodic boundary conditions. If u is a function in $L^2([-1, 1])$ we can compute the constants

$$\alpha_k = \frac{\langle u, \psi_k \rangle}{\|\psi_k\|^2} = \frac{1}{2} \int_{-1}^{1} u(x) e^{-ik\pi x} \, dx. \qquad (8.1)$$

periodic-fourier-coeff

(Recall that in Example 6.25 we computed $\|\psi_k\|^2 = 2$.) The constants α_k in (8.1) are called the ***periodic Fourier coefficients*** of the function u. For any N the sum

$$u_N = \sum_{k=-N}^{N} \alpha_k e^{uk\pi x}$$

does best in approximating u using the function $\psi_{-N}, \ldots, \psi_N$. Since $\|u\|$ is finite, it follows from Bessel's inequality (7.4) that the sum

$$\sum_{k=-\infty}^{\infty} \alpha_k e^{ik\pi x}$$

converges. This sum is called the ***periodic Fourier series*** of the function u.

Example 8.1. *Consider the function $u(x) = x$. The corresponding Fourier coefficients were computed in Example 6.25 to be*

$$a_0 = 0 \quad \text{and}$$
$$a_k = \frac{(-1)^k}{k\pi i} \quad \text{if } k \neq 0.$$

Thus the Fourier series for u is

$$\sum_{k\neq 0} \frac{(-1)^k}{k\pi i} e^{ik\pi x} = \sum_{k=1}^{\infty} \frac{2(-1)^k}{k\pi} \sin(k\pi x).$$

We know that the Fourier series of a function converges to some thing. The remaining sections of this chapter address the question: Does the Fourier series converge to the original function itself?

Exercise 8.2. *Find the periodic Fourier series for the following functions. (Always we are working in $L^2([-1, 1])$.*

1. $u(x) = x^2$
2. $u(x) = |x|$
3. $u(x) = |x|^{1/2}$

4. $u(x) = |\sin(\pi x)|$

Exercise 8.3. *While it is convenient to work on the interval* $[-1, 1]$*, we sometimes need to consider the domain* $[-L, L]$*. Notice that if* $-1 \leq x \leq 1$*, then* $-L \leq xL \leq L$*. Thus if we make the change of variables* $y = xL$ *we can convert between the interval* $[-1, 1]$ *and the interval* $[-L, L]$*.*

1. *Show that the orthogonal functions become* $\psi_k(y) = e^{\frac{ik\pi y}{L}}$*. What is the norm of* $\psi_k(y)$ *in* $L^2([-L, L])$*?*

2. *Show that the formula for the Fourier coefficients becomes*

$$\alpha_k = \frac{1}{2L} \int_{-L}^{L} u(y) e^{-\frac{ik\pi y}{L}}\, dy$$

3. *Find the Fourier series on the interval* $[-L, L]$ *for the function* $u(x) = x$*.*

4. *Let a be some fixed, small number. Find the Fourier series on the interval* $[-L, L]$ *for the function*

$$u(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ 0 & \text{else.} \end{cases}$$

8.2 Notions of convergence

• Move up?

In the previous chapter we discussed a notion of convergence called convergence in norm. There are, in fact, two possible notions of convergence for series.

Pointwise convergence We say that a sequence of functions u_N ***converges pointwise*** to u if for each x, the numbers $u_N(x)$ converge to the number $u(x)$.

Convergence in norm We say that a sequence of functions u_N ***converge in norm*** to u if $\|u_N - u\| \to 0$ as $N \to \infty$. (In this course the norm refers to the L^2 norm, unless otherwise specified.)

Pointwise convergence treats each input x independently. If a sequence of functions u_n converges pointwise to function u, then we know that convergence happens for each x value, but the rate of convergence can be very different for different x values. This leaves open the possibility for very interesting behavior; see 8.4.

Convergence in norm, on the other hand, treats the whole domain all at once. Since the L^2 norm is defined using an integral, convergence in norm means that "overall" the functions u_n are getting close to the function u. This leaves open, however, differences that have small integral.

rgence-counterexamples **Example 8.4.** *In this example we consider two sequences of functions that highlight the difference between pointwise convergence and convergence in norm. For simplicity we work in $L^2([0,1])$.*

1. *Consider the functions $u_n(x) = n^2 x e^{-nx}$. It is easy to see that, for any fixed value of $x \in [0,1]$, we have*

$$\lim_{n\to\infty} u_n(x) = 0.$$

Thus we say that the sequence of functions u_n converges pointwise to the function $u(x) = 0$.

Notice, however, that the maximum of each function u_n occurs at $x = 1/n$, and that $u_n(1/n) = n/e$. Thus

$$\max_{[0,1]} |u_n(x) - u(x)| = \frac{n}{e} \to \infty.$$

Thus even though the functions u_n converge to zero for each fixed x, the maximum distance between u_n and zero is in fact growing!

We can also compute the L^2 norm of $u_n - u$ to be

$$\begin{aligned} \|u_n - u\|^2 &= \int_0^1 n^4 x^2 e^{-2nx}\, dx \\ &= \frac{n}{4} - \frac{1}{4}(2n^3 + 2n^2 + n)e^{-2n} \end{aligned}$$

Thus $\|u_n - u\|^2 \to \infty$ as $n \to \infty$ and thus the sequence of functions u_n does *not converge to u in norm.*

2. *Consider now the functions* $u_n(x) = ne^{-n^3x}$. *Clearly* $u_n(0) = n$, *which diverges as* $n \to \infty$. *However, if* $x > 0$ *then* $u_n(x) \to 0$ *as* $n \to \infty$. *Thus, while the sequence* u_n *does not converge pointwise to* $u(x) = 0$, *it is true that we have pointwise convergence for "most" values of* x.

   We can also compute

$$\begin{aligned}\|u_n - u\|^2 &= \int_0^1 n^2 e^{-2n^3x}\,dx \\ &= \frac{e^{-2n^3} - 1}{2n^3}\end{aligned}$$

   from which we see that u_n *does converge in norm to the zero function.*

Exercise 8.5. *Consider the functions* $v_k(x) = x^k$ *in* $L^2([0,1])$.

1. *Make a plot of the first three or four functions* v_k.
2. *Show that* v_k *converges in norm to* $v = 0$.
3. *Does the sequence converge pointwise to some function? Explain.*

Exercise 8.6. *Consider the functions* w_k *in* $L^2([0,1])$ *defined by*

$$w_k(x) = \begin{cases} \sqrt{k} & \text{if } 0 \le x \le \frac{1}{k} \\ 0 & \text{otherwise.} \end{cases}$$

1. *Make a plot of a typical function* w_k. *What is the "area under the curve" of each function? What happens to the area as* $k \to \infty$*?*
2. *Explain why the sequence converges pointwise to the function* w *given by*

$$w(x) = \begin{cases} \text{undefined} & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. *Show that* $\|w_k\| = 1$ *for all* k.

4. *Explain how it is possible/reasonable/etc for a function that converges pointwise to zero at all points but one to still have norm equal to* 1. *What's going on here?*

Exercise 8.7. *Write a short paragraph describing the difference between convergence in norm and pointwise convergence.*

At the end of this chapter, we give conditions under which the Fourier series converge in norm, and conditions under which the Fourier series partial sum converges pointwise. Before we present these results, we first introduce some technical tools.

8.3 Approximate identities

Frequently in our physics courses we encounter the concept of a "point source." It is convenient, for example, to consider a mass (or a charge) to be at a particular location. Mathematically, however, point sources are somewhat difficult as the mass (or charge) density corresponding to such an object cannot be a function in the usual sense. To see this, suppose that we want to discuss an object of unit mass located at the origin. The mass density function for such an object would be zero at all points except at $x = 0$. Since integration typically ignores what happens at a single point, we might expect that the integral of such a mass density function to be zero. This, however, is a contradiction – the integral of the mass density function needs to be equal to the total mass of the object.

The class of mathematical objects that get us out of this pickle are called "distributions." These are generalizations of functions that are not defined pointwise, but are only defined in terms of how they behave under integration. Point sources can be described using the ***Dirac delta distribution***, which is given the symbol δ and is defined by

$$\int_{-\infty}^{\infty} \delta(y)\phi(x-y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^{\infty}(\mathbb{R}).$$

It follows directly from the definition that δ is even (in the sense that $\delta(x) = \delta(-x)$). Using the even property, making a change of variables, and setting $x = 0$, we see that

$$\int_{-\infty}^{\infty} \delta(y)\phi(y)\,dy = \phi(0) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}). \tag{8.2}$$ AltDirac

It is common to interpret (8.2) as meaning that the Dirac distribution $\delta(x)$ "is zero for $x \neq 0$ and infinite when $x = 0$ in such a way that the total size is 1." While this interpretation works well for physical though experiments, it is somewhat difficult to deal with mathematically.

One way to put the Dirac delta distribution on mathematical footing is to introduce is the concept of an "approximate identity." Let Ω be a subset of $\mathbb{R}$. A sequence of functions $K_n(x)$ is an ***approximate identity*** for the domain Ω if

$$\int_\Omega K_n(x)\,dx = 1$$

and

$$\lim_{n\to\infty} \int_\Omega K_n(y)\phi(x-y)\,dy = \phi(x)$$

for all test functions $\phi \in C_0^\infty(\Omega)$. (Here we can view ϕ as having the domain of $\mathbb{R}$, but equal to zero outside Ω.)

AI-step-example **Example 8.8.** *Consider the functions*

$$K_n(x) = \begin{cases} \frac{n}{2} & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

on the domain $\Omega = [-1, 1]$*; see 8.1.*

I claim that K_n *is an approximate identity.*

It is straightforward to compute

$$\int_{-1}^{1} K_n(x)\,dx = \int_{-1/n}^{1/n} \frac{n}{2}\,dx = 1;$$

thus the first property is satisfied.

To see the second property, let ϕ *be a test function that is zero outside some region*

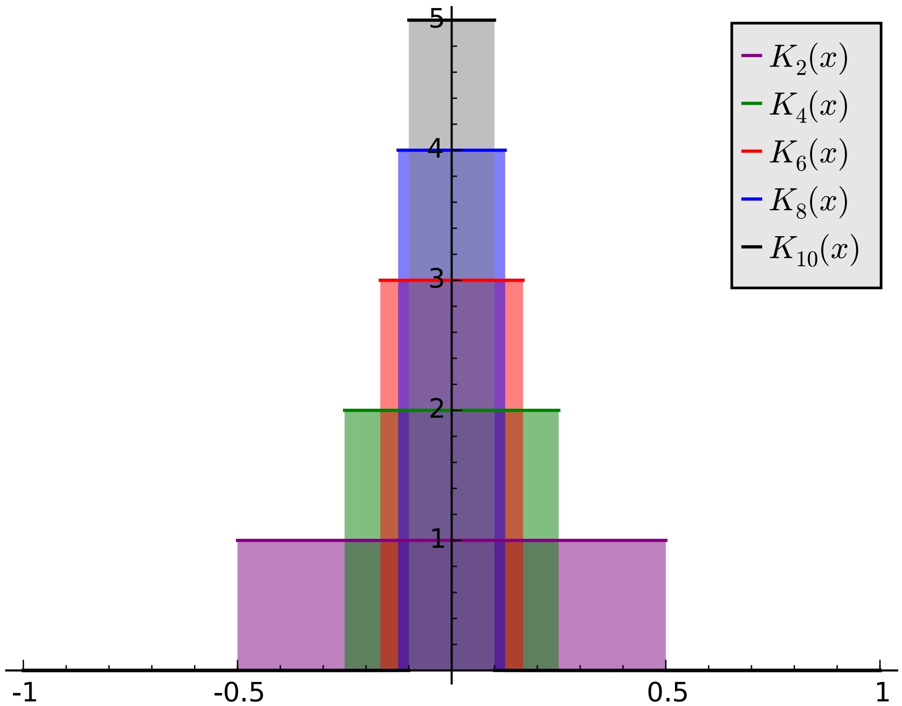

Figure 8.1: The functions K_n for $n = 2, 4, 6, 8, 10$. As n gets large, the functions are concentrated near $x = 0$. Yet each corresponding shaded area, and hence integral, is always 1. Thus in the limit as $n \to \infty$, the functions K_n approximate a point source at the origin.

figAI-step-plot

in $(-1, 1)$ *and define*

$$\Phi(z) = \int_0^z \phi(x-y)\,dy.$$

Notice that $\Phi(0) = 0$ *and thus*

$$\begin{aligned}\int_{-1}^{1} K_n(y)\phi(x-y)\,dy &= \frac{n}{2}\int_{-1/n}^{0} \phi(x-y)\,dy + \frac{n}{2}\int_0^{1/n} \phi(x-y)\,dy \\ &= \frac{1}{2}\left[\frac{\Phi(-1/n)-\Phi(0^-)}{-1/n} + \frac{\Phi(1/n)-\Phi(0^+)}{1/n}\right]\end{aligned}$$

Therefore the Fundamental Theorem of Calculus implies

$$\lim_{n\to\infty}\int_{-1}^{1} K_n(y)\phi(x-y)\,dy = \frac{1}{2}\left[\Phi'(0^-)+\Phi'(0^+)\right] = \frac{1}{2}\left[\phi(x^-)+\phi(x^+)\right] = \phi(x).$$

Notice that we do not actually need ϕ to be a test function – it is enough for ϕ to be piecewise continuous. However, in the case that ϕ is not continuous at x, then the limit is the average of the left and right limits of ϕ.

The approximate identity K_n in 8.8 is not particularly well suited for working with the periodic Fourier series. Thus in the proof of 8.12 we instead make use of the approximate identity

$$D_n(x) = \frac{1}{2}\sum_{k=-n}^{n} e^{ik\pi x},$$

which called the ***Dirichlet kernel***. Plots of D_n for the first few values of n appear in 8.2. Before showing that D_n is an approximate identity, we simply derive some properties of the Dirichlet kernel.

Exercise 8.9.

1. *Show that*

$$D_n(x) = \frac{1}{2}e^{-ik\pi x}\sum_{l=0}^{2l}\left(e^{i\pi x}\right)^l.$$

2. *Use the identity above to show that*

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos(k\pi x) = \frac{\sin\left(\frac{(2n+1)\pi}{2}y\right)}{2\sin\left(\frac{\pi}{2}y\right)} \tag{8.3}$$ DK-trig-identity

and that

$$\int_{-1}^{0} D_n(x)\,dx = \frac{1}{2} = \int_{0}^{1} D_n(x)\,dx. \tag{8.4}$$ DK-integral

3. *Finally, show that that D_n is periodic, with period 2, and that D_n is an even function.*

We now show that D_n is an approximate identity. Fix $\phi \in C_0^\infty([-1,1])$ and fix some

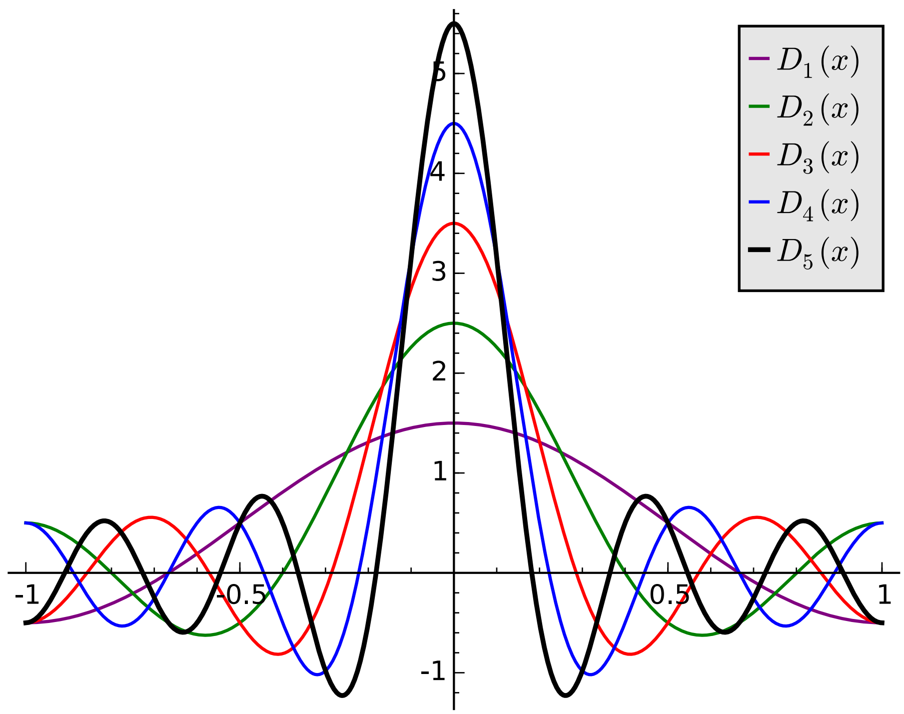

Figure 8.2: Plots of the Dirichlet kernel $D_n(x)$ for several values of n. Notice... fig:DK-plot

$x \in [-1, 1]$. We want to show that

$$\lim_{n\to\infty}\left[\int_{-1}^{1} D_n(y)\phi(x-y)\,dy - \phi(x)\right] = 0. \tag{8.5}$$ DK-claim-ID

Before proceeding, let us make sure that the integral in (8.5) makes sense. Since we are integrating on the interval $-1 \le y \le 1$, the test function ϕ is being evaluated on the interval $x-1 \le x-y \le x+1$. However, since $\phi \in C_0^\infty([-1,1])$ we can view ϕ as a function defined on all of $\mathbb{R}$, with $\phi = 0$ outside of $[-1, 1]$. Thus the integral does indeed make sense.

Using (8.4) and (8.3) we have

$$\begin{aligned}\int_{-1}^{1} D_n(y)\phi(x-y)\,dy - \phi(x) &= \int_{-1}^{1} D_n(y)\left[\phi(x-y)-\phi(x)\right]\,dy \\ &= \int_{-1}^{1} \frac{\phi(x-y)-\phi(x)}{2\sin\left(\frac{\pi}{2}y\right)}\,\sin\left(\frac{(2n+1)\pi}{2}y\right)\,dy.\end{aligned}$$

Our strategy is to view this as the Fourier coefficient for the function

$$\Phi(y) = \frac{\phi(x-y)-\phi(x)}{2\sin\left(\frac{\pi}{2}y\right)} = \frac{\phi(x-y)-\phi(x)}{y}\,\frac{y}{2\sin\left(\frac{\pi}{2}y\right)}.$$

The function $\Phi(y)$ is defined for all $y \neq 0$.

Exercise 8.10. *Use the fact that ϕ is differentiable to show that the limit as $y \to 0$ of $\Phi(y)$ exists. Conclude that $\Phi(y)$ is a piecewise continuous function with a removable discontinuity at $y = 0$.*

We now proceed by invoking 7.9. The functions

$$v_n = \sin\left(\frac{(2n+1)\pi}{2}y\right)$$

form an orthogonal collection in $L^2([-1,1])$ with $\|v_n\| = 1$. Thus (8.5) is equivalent to

$$\lim_{n\to\infty} \langle \Phi, v_n \rangle = 0,$$

which is precisely the statement of 7.9. Thus we conclude that D_n is in fact an approximate identity.

8.4 Convergence for periodic Fourier series

hm:FS-converge-in-norm **Theorem 8.11** (Convergence in norm for Fourier series)**.** *Suppose u is a function in $L^2([-1,1])$. Then the sequence of functions u_N given by*

$$u_N = \sum_{k=-N}^{N} \alpha_k \psi_k \tag{8.6}$$ FS-partial-sum

converges in norm to u.

The proof of 8.11 is somewhat complicated, and is thus reserved for one of the "excursions"; see 33.

The explorations in the exercises at the end of the previous chapter suggest that the functions in (8.6) converge pointwise at all but a few exceptional points. It turns out that that this is true, provided we make some assumptions about the function u. (These assumptions are sometimes called "Dirichlet conditions" – not to be confused with "Dirichlet boundary conditions.") The remainder of this chapter is devoted to proving the following version of of the pointwise convergence theorem.

-pointwise-convergence **Theorem 8.12** (Pointwise convergence for Fourier series)**.** *Suppose*

$$u \in L^2_{\mathrm{PC}}([-1,1]) \quad \textit{and} \quad u' \in L^2_{\mathrm{PC}}([-1,1]).$$

(The second condition implies that at each point both the left and right derivatives of u exists, meaning that at each $x \in [-1,1]$ the limits

$$f'(x^-) = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} \quad \textit{and} \quad f'(x^+) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

exist.) Then for each $x \in [-1,1]$ the sequence $u_N(x)$ converges to

$$\frac{1}{2}\left(f(x^-) + f(x^+)\right) = \frac{1}{2}\left(\lim_{y\to x^-} f(y) + \lim_{y\to x^+} f(y)\right).$$

The remainder of this section is devoted to the proof of 8.12. Suppose that $u, u' \in L^2_{\mathrm{PC}}([-1,1])$. Let u_N be the partial sum given by (8.6) and fix some $x \in [-1,1]$.

Our goal is to show that

$$\lim_{N\to\infty} \left[u_N(x) - \frac{1}{2}\left(u(x^-) + u(x^+)\right) \right] = 0.$$

Before we begin, we need to make technical clarification. For points x that are strictly between -1 and 1, it is clear what we mean by $u(x^-)$ and $u(x^+)$. However, if $x = 1$, then the right limit $u(x^+)$ does not really make sense. The way to handle this situation is to extend u periodically to be defined for all real numbers. We then define $u(1^+)$ and $u(-1^-)$ using these periodic extensions.

Example 8.13. *Consider the function* $u(x) = x$ *with domain* $[-1, 1]$. *When extended periodically to be defined on the whole real line, the function is no longer continuous. Rather, it has jump discontinuities at* $x = 2k + 1$ *for all integers* k*; see 8.3.*

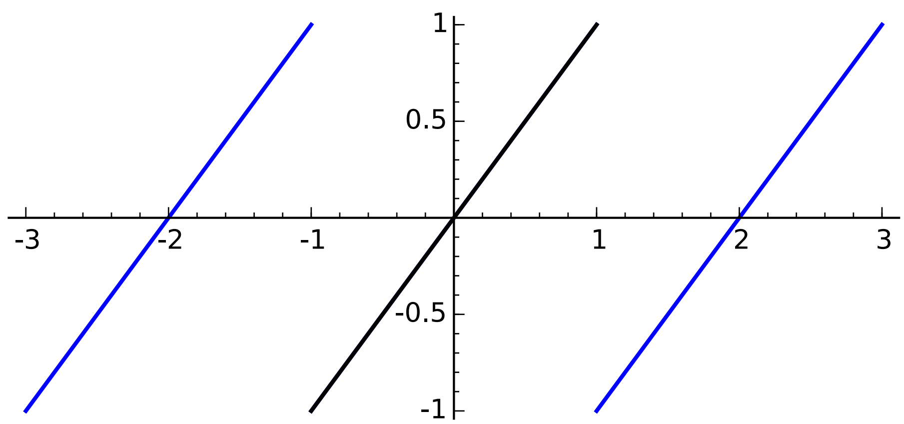

Figure 8.3: The periodic extension of the function $u(x) = x$, known as the "sawtooth wave." Notice that $u(1^+) = -1$, while $u(1^-) = 1$. fig:sawtooth-plot

Exercise 8.14. *Consider the function*

$$u(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

Draw a picture of the periodic extension of u. What is $u(1^+)$*? What is* $u(1^-)$*?*

The following is "essentially obvious" but nevertheless important.

ex:periodic-integral **Exercise 8.15.** *Draw a picture that illustrates the following fact: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period 2. Then for any real number a we have*

$$\int_a^{a+2} f(x)\,dx = \int_{-1}^{1} f(x)\,dx.$$

We now proceed to prove 8.12 using an argument that is similar to the argument used to show that the Dirichlet kernel D_n is in fact an approximate identity. The argument is made a bit more complicated by the fact that $u(x)$ is not a test function, but otherwise follows the exact same pattern.

We begin with the following.

ial-sum-is-convolution **Exercise 8.16.** *Show that the function $u_N(x)$ given by* (8.6) *can be written*

$$u_N(x) = \int_{-1}^{1} \frac{1}{2} \sum_{k=-N}^{N} u(y) e^{ik\pi(x-y)}\,dy.$$

Exercise 8.16 implies that

$$u_N(x) = \int_{-1}^{1} D_N(y) u(x+y)\,dy.$$

Thus using (8.4) we have

$$\underbrace{u_N(x) - \frac{1}{2}\left(u(x^-) + u(x^+)\right)}_{I}$$
$$= \underbrace{\int_{-1}^{0} D_N[u(x+y) - u(x^-)]\,dy}_{I^-} + \underbrace{\int_{0}^{1} D_N[u(x+y) - u(x^+)]\,dy}_{I^+}.$$

Using (8.3), we see that

$$I^- = \int_{-1}^{0} \frac{u(x+y)-u(x^-)}{2\sin\left(\frac{\pi}{2}y\right)}\left[\cos\left(\frac{\pi}{2}y\right)\sin\left(N\pi y\right) + \sin\left(\frac{\pi}{2}y\right)\cos\left(N\pi y\right)\right] dy,$$

$$I^+ = \int_{0}^{1} \frac{u(x+y)-u(x^+)}{2\sin\left(\frac{\pi}{2}y\right)}\left[\cos\left(\frac{\pi}{2}y\right)\sin\left(N\pi y\right) + \sin\left(\frac{\pi}{2}y\right)\cos\left(N\pi y\right)\right] dy.$$

If we define the functions

$$F(y) = \begin{cases} \dfrac{u(x+y)-u(x^-)}{2\sin\left(\frac{\pi}{2}y\right)}\cos\left(\frac{\pi}{2}y\right) & \text{if } y < 0 \\ \\ \dfrac{u(x+y)-u(x^+)}{2\sin\left(\frac{\pi}{2}y\right)}\cos\left(\frac{\pi}{2}y\right) & \text{if } y > 0 \end{cases}$$

and

$$G(y) = \begin{cases} \dfrac{u(x+y)-u(x^-)}{2} & \text{if } y < 0 \\ \\ \dfrac{u(x+y)-u(x^+)}{2} & \text{if } y > 0 \end{cases}$$

then we have

$$I = \langle F(y), \sin(N\pi y)\rangle + \langle G(y), \cos(N\pi y)\rangle.$$

Exercise 8.17. *I claim that F and G are in $L^2([-1,1])$. Explain why this fact, together with 7.9, implies the desired convergence of the Fourier series.*

We now verify the claim that F and G are in $L^2([-1,1])$. First, we observe that u being piecewise continuous implies that G is bounded and thus in $L^2([-1,1])$.

Second, recall that we are assuming that the left and right derivatives of u exist at input x. This implies that

$$\frac{u(x+y)-u(x^-)}{2\sin\left(\frac{\pi}{2}y\right)} = \frac{u(x+y)-u(x^-)}{y}\,\frac{y}{2\sin\left(\frac{\pi}{2}y\right)}$$

is bounded for $-1 \le y < 0$. Hence $F(y)$ is bounded for $-1 \le y < 0$; an analogous argument implies that $F(y)$ is bounded for $0 < y \le 1$. Thus $F \in L^2([-1,1])$.