Chapter 7

Complete orthogonal collections of vectors

ch:completeness

7.1 Bases and finite dimensionality

• Combine this with best approximation section

One of the most important (and useful) features of $\mathbb{R}^n$ is that there is a short list of vectors, called the *standard basis vectors*, having two important properties:

- Any other vector in $\mathbb{R}^n$ can be uniquely written as a linear combination of the standard basis vectors.
- The standard basis vectors are mutually orthogonal.

These standard basis vectors are

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

In particular, if $\mathbf{u} \in \mathbb{R}^n$, then there exists unique numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n.$$

The numbers α_k are often called the " components (or coordinates) of $\mathbf{u}$ relative to the standard basis vectors."

Since the standard basis vectors are orthogonal, there is an easy formula for computing the coordinates:

$$\alpha_k = \frac{\mathbf{u} \cdot \mathbf{e}_k}{\mathbf{e}_k \cdot \mathbf{e}_k}.$$

In retrospect, this formula is not too surprising – it simply states that the combination of the standard basis vectors that best approximates $\mathbf{u}$... is $\mathbf{u}$ itself!

The standard basis vectors are not the only collection of vectors having the properties listed above.

Example 7.1. *Consider the following vectors in $\mathbb{R}^3$:*

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

It is easy to check that these three vectors are all orthogonal to one another.

To see that any vector $\mathbf{u}$ *can be expressed uniquely as a linear combination of these three vectors, we consider a generic vector*

$$\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and try to find coefficients $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{u}.$$

This reduces to a system of three equations for three unknowns, which can be solved *explicitly. The result is that there is only one solution, namely*

$$\mathbf{u} = \underbrace{\left(\frac{a}{3} + \frac{b}{3} + \frac{c}{3}\right)}_{\alpha_1} \mathbf{v}_1 + \underbrace{\left(\frac{a}{6} - \frac{b}{3} + \frac{c}{6}\right)}_{\alpha_2} \mathbf{v}_2 + \underbrace{\left(\frac{a}{2} - \frac{c}{2}\right)}_{\alpha_3} \mathbf{v}_3.$$

We can also verify that the coefficients $\alpha_1, \alpha_2, \alpha_3$ *are given by the formula*

$$\alpha_k = \frac{\mathbf{u} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}. \tag{7.1}$$

CoefficientsOfOrthogona

For a general vector space, we say that a *finite* list of vectors $v_1, \ldots, v_n$ forms a ***basis*** for the vector space if every vector u can be written uniquely as a linear combination of $v_1, \ldots, v_k$. It is important to emphasize that there are two aspects of this definition: there must *exist* a combination of the vectors that forms u, and this combination must be *unique*.

If a vector space admits a basis, then it is called ***finite dimensional*** and the ***dimensional*** is defined to be the number of vectors in the basis. It is a fact from linear algebra, where finite dimensional vector spaces are studied extensively, that all bases of a finite dimensional vector space have the same number of vectors.

Warning 7.2. *There are a number of variations on the definition of a basis. Here are two such variations to consider.*

1. *Many linear algebra texts define a basis to be a list of vectors that are linearly independent and that span the entire vector space. This definition is equivalent to the definition used in these notes.*
2. *In the case of infinite-dimensional vector spaces, it is possible to extend the definition of basis to include non-finite (and, indeed, uncountable) collections of vectors. A number of subtle issue arise when working with such bases; the roots of these issues is related to the phrase "linear combination."*

We emphasize that in these notes, we use the convention that "linear combination" means a finite linear combination.

It is important that the vectors $v_1, \dots, v_n$ do not need to be orthogonal in order to comprise a basis. Indeed, we do not even need to have an inner product present in order for the definition to make sense! An example of a basis for $\mathbb{R}^3$ consisting of non-orthogonal vectors appears in 7.3. However, if we do have an inner product present, then the formula (7.1) only works if the vectors are orthogonal!

ex:BasisOfR3 **Exercise 7.3.** *Show that the vectors*

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$ *by showing that a generic vector* $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *can be uniquely expressed as a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$*. Show that the formula* (7.1) *does not give the correct coefficients.*

Most of the vector spaces that we study in the course do have inner products, but are not finite dimensional.

2-infinite-dimensional **Example 7.4.** *Recall the vector space* $l^2(\mathbb{N})$ *that is introduced in 5.6. For each natural number* k*, let* E_k *be the sequence that is all zero, except that the* k^{th} *number is one. Thus*

$$\begin{aligned} E_1 &= \{1, 0, 0, 0, \dots\} \\ E_2 &= \{0, 1, 0, 0, \dots\} \\ E_3 &= \{0, 0, 1, 0, \dots\} \\ &\textit{etc.} \end{aligned}$$

It is easy to see that there cannot be a finite list of sequences such that all of these sequences can be constructed by linear combinations. Thus $l^2(\mathbb{N})$ *is infinite dimensional.*

We can define an inner product on $l^2(\mathbb{N})$ *as follows. If* $A = \{a_k\} = \{a_1, a_2, a_3, \dots\}$

and $B = \{b_k\}$*, then we set*

$$\langle A, B\rangle = \sum_{k=1}^{\infty} a_k b_k .$$

Notice that with this inner product, the sequences E_k *are all orthogonal to one another.*

Notice also that if a sequence $A = a_k$ *is in* $l^2(\mathbb{N})$*, then we can express* A *in terms of the sequences* E_k *by*

$$A = \sum_{k=1}^{\infty} a_k E_k . \tag{7.2}$$

l2-example-sum

We need to be somewhat careful with how we interpret the summation in (7.2)*, however, because technically this is the sum of a* list of sequences*, not a list of numbers.*

The example in Example 7.4 illustrates the need to develop a few technical tools, so that we can make sense of expressions such as (7.2). In particular, we need:

- to be able to make sense of infinite sums of vectors in general vector spaces, and
- to find a condition under which an infinite list of orthogonal vectors "behaves like a basis" in the sense that each vector can be constructed by an infinite sum of the vectors.

These issues are the subject of the next two sections.

7.2 Infinite sums in vector spaces

In our first-year calculus course, we learn that the infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

converges to 1. Although we commonly say that the "sum is equal to 1," what we actually mean is that the "limit of the partial sums is 1." This is expressed mathematically as

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{2^k} = 1.$$

An equivalent way to show that the sum converges to 1 is to show that "the difference between the partial sum and 1 has a limit of zero." We can write this version as

$$\lim_{N\to\infty} \left| \sum_{k=1}^{N} \frac{1}{2^k} - 1 \right| = 0.$$

The advantage of this latter way of writing things is that the quantity inside the absolute value is a *finite* sum, and can also be easily generalized to the case of vectors.

Motivated by the above discussion, we make the following definition. Suppose $v_1, v_2, v_3, \ldots$ is an infinite list of vectors, taken from an inner product space. We say that ***the sum of these vectors converges in norm to*** v, and write

$$\sum_{k=1}^{\infty} v_k = v,$$

if

$$\lim_{N\to\infty} \left\| \sum_{k=1}^{N} v_k - v \right\| = 0.$$

Here the phrase "in norm" reminds us that we are using the norm (which comes from the inner product) to measure the difference between the partial sum and the limiting vector v.

The following example is adapted from Walter Rudin's book *Principles of Mathematical Analysis* (1976).

L2-ConvergenceExample

Example 7.5. *Consider the vector space $L^2([-1, 1])$ and define the sequence of functions v_k by*

$$v_k(x) = \frac{x^2}{(1 + x^2)^k}, \qquad k = 0, 1, 2, 3, \ldots.$$

I claim that the sum of these functions converges in norm to the function $v(x) = 1 + x^2$.

To see this, we compute the partial sum using geometric series:

$$\begin{aligned}\sum_{k=0}^{N} v_k &= x^2 \sum_{k=0}^{N} \left(\frac{1}{1+x^2}\right)^k \\ &= x^2 \frac{1-\left(\frac{1}{1+x^2}\right)^{N+1}}{1-\left(\frac{1}{1+x^2}\right)} \\ &= (1+x^2) - \frac{1}{(1+x^2)^N}.\end{aligned}$$

Thus

$$\left\| \sum_{k=0}^{N} v_k - v \right\|^2 = \int_{-1}^{1} \frac{1}{(1+x^2)^{2N}}\, dx.$$

As $N \to \infty$ the integrand tends to zero for all values of x except $x = 0$. Thus

$$\lim_{N\to\infty} \left\| \sum_{k=0}^{N} v_k - v \right\| = 0.$$

The convergence can be visualized using 7.1.

This particular list of functions is actually interesting for another reason. Notice that $v_k(0) = 0$ for each k. Thus you might expect that the limiting function to have output 0 when $x = 0$. However, this is not the case, as $v(0) = 1$. This is an example of an important phenomenon – there can be "jumps" in the limiting process. These "jumps" are discussed in more detail below.

7.3 Complete collections

Suppose we have an infinite list of orthogonal vectors $v_1, v_2, v_3, \ldots$ in an inner product space. Suppose also that u is some other vector. For each finite list $v_1, v_2, \ldots, v_N$ we can construct the best approximation of u by the finite list. This

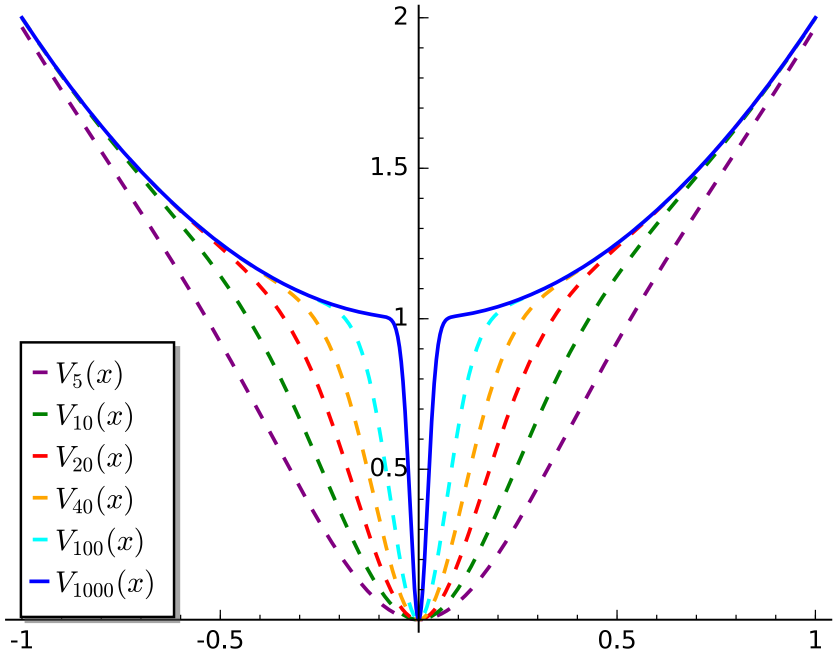

Figure 7.1: Plots of several partial sums $V_N = \sum_{k=1}^{N} v_k$. The graphic illustrates that the sum converges to an 'indented' version of $v(x) = 1 + x^2$. fig:rudins-example

approximation, which we give the symbol u_N, is given by

$$u_N = \sum_{k=1}^{N} \alpha_k v_k, \qquad \text{where} \qquad \alpha_k = \frac{\langle u, v_k \rangle}{\|v_k\|^2}. \tag{7.3}$$ ApproximationPartialSum

The main question of this section is: Under what circumstances does u_N converge in norm to u as $N \to \infty$? This question is a technical way of asking: "How do I know if my list of vectors $v_1, v_2, v_3, \ldots$ has "enough vectors" so that I can build any other vector from the vectors in the list?"

Before we address the question of when an orthogonal list has enough vectors, we first establish two useful facts.

Theorem 7.6 (Bessel's inequality)**.** *Suppose $v_1, v_2, v_3, \ldots$ in an infinite list of orthogonal vectors in an inner product space. Let u be any vector, and let α_k be as in*

(7.3). *Then*

$$\sum_{k=1}^{\infty} \|\alpha_k v_k\|^2 \leq \|u\|^2. \tag{7.4}$$

BesselsInequality

Proof. The proof of the theorem is simple. From (6.10), we know that $\|u_N\|^2 \leq \|u\|^2$ and thus each partial sum is bounded. Since the terms in the series are non-negative, this implies that the series converges. Taking the limit as $N \to \infty$ yields the desired result. □

Example 7.7. *Recall the periodic sine functions $w_k(x) = \sin(\pi k x)$ appearing in 6.17 and also recall the function $u(x) = x$ from that same example. We previously computed $\|w_k\| = 1$ and $\alpha_k = \langle w_k, u\rangle = \frac{2(-1)^{k+1}}{k\pi}$. Since $\|u\|^2 = 2/3$, Bessel's inequality implies that*

$$\sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} \leq \frac{2}{3},$$

which we may also write

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2}{6}.$$

Bessel's inequality also holds for an infinite list of orthogonal vectors indexed by the integers.

Exercise 7.8. *Let $\psi_k(x) = e^{ik\pi x}$ be orthogonal functions in $L^2([-1,1])$ that were discussed in Example 6.25, and let*

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

1. *Compute $\|u\|^2$.*

2. *Compute $\langle u, \psi_k\rangle$*

3. *Write down the formula for the sum*

$$u_N(x) = \sum_{k=-N}^{N} \alpha_k \psi_k, \quad \text{where} \quad \alpha_k = \frac{\langle u, \psi_k\rangle}{\|\psi_k\|^2}.$$

4. *Make a plot of $u_5(x)$ and compare with the plot of $u(x)$.*

5. *Write down the inequality arises from applying Bessel's inequality to this situation.*

In the first-year calculus course we learn that if a series of non-negative numbers converges, it must be that the list of numbers tends to zero. This, together with Bessel's inequality, implies the following.

thm:Riemann-Lebesgue **Theorem 7.9** (Riemann-Lebesgue lemma)**.** *Suppose $v_1, v_2, v_3, \ldots$ is an infinite list of orthogonal vectors in an inner product space. Let u be any vector, and let α_k be as in* (7.3)*. Then*

$$\lim_{k\to\infty} \|\alpha_k v_k\|^2 = 0$$

and thus

$$\lim_{k\to\infty} \frac{\langle u, v_k\rangle}{\|v_k\|} = 0.$$

Bessel's inequality (7.4), together with the Riemann-Lebesgue Lemma, implies that the sum

$$\sum_{k=1}^{\infty} \alpha_k v_k = \lim_{N\to\infty} u_N$$

converges to some vector – let's call this vector u_∞. Furthermore, the norm of the limiting vector u_∞ is not greater than the norm of u. (It might be helpful to revisit Exercise 6.10 at this moment.)

However, the limiting vector u_∞ might not actually be equal to u — the reason is that we might not have "enough" vectors v_k, rather there might be "some direction" that is "missing." The following theorem tells us that in fact we have equality in (7.4) precisely when we have "enough" vectors v_k.

ConvergenceAndParseval **Theorem 7.10.** *Suppose $v_1, v_2, v_3, \ldots$ in an infinite list of orthogonal vectors in an inner product space. Let u be any vector, and let u_N and α_k be as in* (7.3)*. Then*

$$\lim_{N\to\infty} \|u_N - u\| = 0$$

if and only if

$$\sum_{k=1}^{\infty} \|\alpha_k v_k\|^2 = \|u\|^2. \tag{7.5}$$ ParsevalsIdentity

Proof. The proof of this theorem follows from the identity established in Exercise 6.11:

$$\|u\|^2 = \|u_N\|^2 + \|u^{\perp}\|^2,$$

which we re-write as

$$\|u - u_N\|^2 = \|u\|^2 - \|u_N\|^2.$$

Using Exercise 6.10 we obtain

$$\|u - u_N\|^2 = \|u\|^2 - \sum_{k=1}^{N} \|\alpha_k v_k\|^2.$$

The limit on the left is zero exactly when the limit on the right is zero, which is the statement of the theorem. □

When it holds, the identity (7.5) is called ***Parseval's identity***.

We conclude this section with the following.

arsevalImpliesComplete **Theorem 7.11.** *Suppose $v_1, v_2, v_3, \ldots$ in an infinite list of orthogonal vectors in an inner product space such that* (7.5) *holds for any vector u. Then the only vector w satisfying $\langle w, v_k \rangle$ for all k is $w = 0$.*

Lists of orthogonal vectors v_k with the property that the only $w = 0$ is orthogonal to all of them are called ***complete***. Thus the combination of 7.10 and 7.11 shows that if our orthogonal collection contains "enough vectors to construct any vector" then it is complete.