CHAPTER 1

Review: The simple harmonic oscillator

ch:SHO-eigenstuff

1.1. The simple harmonic oscillator and scaling solutions

Recall the simple harmonic oscillator

$$m\frac{d^2u}{dt^2} = -ku. \tag{1.1}$$ SHO

Here u represents the displacement from equilibrium of some oscillator, and (1.1) is simply Newton's formula $ma = F$ with the force being given by Hooke's formula $F = -ku$. It is common to set $\omega^2 = k/m$, so that (1.1) becomes

$$\frac{d^2u}{dt^2} = -\omega^2 u. \tag{1.2}$$ SHO2

Introducing $v = \frac{du}{dt}$ we can write (1.2) as the first-order system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.3}$$ SHO-FirstOrder

Letting

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0, \end{pmatrix} \tag{1.4}$$ SHO-FirstOrderPieces

we can write (1.3) as

$$\frac{d}{dt}\mathbf{u} = M\mathbf{u}. \tag{1.5}$$ SHO-CompactForm

We proceed by looking for solutions to (1.5) of the form

$$\mathbf{u} = A(t)\,\mathbf{u}_*, \tag{1.6}$$ SHO-ScalingAnsatz

where $\mathbf{u}_* = \begin{pmatrix} u_* \\ v_* \end{pmatrix}$ is a constant vector. Solutions of the form (1.6) are called ***scaling solutions*** because they are simply rescaled as time progresses.

Plugging (1.6) in to (1.3) we see that in order to be a solution we must have

$$\frac{1}{A}\frac{dA}{dt}\mathbf{u}_* = M\mathbf{u}_*.$$

Since the right side of this equation is time independent, there must be some constant λ such that

$$\frac{1}{A}\frac{dA}{dt} = \lambda \tag{1.7}$$

SHO-ScalingODE

and

$$M\mathbf{u}_* = \lambda \mathbf{u}_*. \tag{1.8}$$

SHO-EigenvalueProblem

The equation (1.8) is called the *eigenvalue problem* for matrix M.

Important Point 1.1. *Scaling solutions take the form $A(t)\mathbf{u}_*$ where*

- $\mathbf{u}_*$ *is an eigenvector of M and*
- *the amplitude function $A(t)$ satisfies some differential equation depending on the corresponding eigenvalue λ.*

Direct computation (see Exercise 1.1) shows that the matrix M appearing in (1.4) has eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$. The corresponding scaling solutions are

$$\mathbf{u}_1(t) = e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2(t) = e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}. \tag{1.9}$$

SHO-ComplexSolutions

1.2. Linearity and superposition

We now recall from our differential equations course an important concept: linearity. A differential equation is called ***linear*** if the following property holds:

- If u and v are solutions and α, β are numbers, then $\alpha u + \beta v$ is also a solution.

Using this property, we can construct new solutions to a linear equation from old solutions; this is sometimes called the "superposition" principle.

By direct computation we can easily verify that both (1.1) and (1.3) are linear differential equations. Thus we may use the two scaling solutions in (1.9) to construct more solutions to (1.5). For any complex numbers α_1 and α_2, the function

$$\mathbf{u}(t) = \alpha_1 e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \alpha_2 e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix} \tag{1.10}$$

SHO-ComplexGeneralSolu

is also a solution to (1.5).

In the differential equations course, you studied the *initial value problem* for ordinary differential equations. The initial value problem for the simple harmonic oscillator (in first-order form) consists of the differential equation (1.5), together with the initial condition $\mathbf{u}(0) = \mathbf{u}_0$, where

$$\mathbf{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

is some fixed vector.

In the differential equations course, we addressed the initial value problem by finding a ***general solution***, which is a solution to the differential equation having free constants that can be chosen in order to ensure that the initial condition is satisfied. In Exercise 1.2 you verify that $\mathbf{u}$ given by (1.10) is a general solution to (1.5).

In that exercise you see that the "reason" that (1.10) is a general solution is that there are "enough" free parameters to ensure that any initial condition can be satisfied. Furthermore, that there are "enough" solutions is equivalent to there being "enough" eigenvectors to build any vector out of eigenvectors. If any vector can be constructed from combinations of eigenvectors, then we say that the list of eigenvectors is ***complete***. Thus one way to find a general solution to a linear differential equation is to find complege list scaling solutions and then use the superposition principle.

Important Point 1.2. *If an equation is linear, then one can obtain more solutions by using linearity to combine scaling multiple solutions.*

If any solution can be constructed this way, then we say that the list of scaling solutions is complete.

The completeness of the list of scaling solutions is equivalent to the completeness of corresponding list of eigenvectors.

1.3. Complex and real solutions

You might be concerned that the solutions in (1.9) are complex-valued, while the differential equation (1.3) only involves real numbers. We can, however, obtain real solutions by careful choices of α_1 and α_2 in (1.10). To do this we use the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.11}$$ EulerIdentity

which implies that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

For example, if we choose $\alpha_1 = 1$ and $\alpha_2 = 1$ in (1.10), then we have

$$\mathbf{u} = e^{i\omega t}\begin{pmatrix}1\\ i\omega\end{pmatrix} + e^{-i\omega t}\begin{pmatrix}1\\ -i\omega\end{pmatrix} = \begin{pmatrix}2\cos(\omega t)\\ -2\omega\sin(\omega t)\end{pmatrix}.$$

This issue is explored in more detail in Exercise 1.3. For now, we note the following.

Important Point 1.3. *If we obtain complex-valued scaling solutions we can use linearity to find real-valued solutions by carefully choosing the constants in the general solution and using the Euler identity* (1.11).

Exercises

Ex:SHO-Eigenstuff **Exercise 1.1.** Show that the eigenvalues of the matrix M in (1.4) are $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, and that the corresponding eigenvectors are

$$\begin{pmatrix}1\\ i\omega\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\ -i\omega\end{pmatrix}.$$

Use these eigenvalues and eigenvectors to show obtain the scaling solutions (1.9) to (1.5).

x:SHO-general-solution **Exercise 1.2.**

(1) Show that for any vector $\mathbf{u}_0 = \begin{pmatrix}u_0\\ v_0\end{pmatrix}$ we can choose α_1 and α_2 such that

$$\alpha_1\begin{pmatrix}1\\ i\omega\end{pmatrix} + \alpha_2\begin{pmatrix}1\\ -i\omega\end{pmatrix} = \begin{pmatrix}u_0\\ v_0\end{pmatrix}.$$

(2) Explain how to use the previous part in order to show that the initial value problem for (1.5) has a solution for all initial conditions.

ex:SHO-real-solutions **Exercise 1.3.** The scaling solutions given in (1.9) involve complex numbers. Here we see how these nevertheless give rise to real-valued solutions.

(1) Find the solution $\mathbf{u}_c$ to the initial value problem

$$\frac{d}{dt}\mathbf{u} = M\mathbf{u} \qquad \mathbf{u}(0) = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

Use the Euler identity (1.11) to show that $\mathbf{u}_c$ is in fact real-valued.

(2) Find the solution $\mathbf{u}_s$ to the initial value problem

$$\frac{d}{dt}\mathbf{u}_s = M\mathbf{u}_s \qquad \mathbf{u}_s(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{u}_s$ is also fact real-valued.

(3) Explain why

$$\mathbf{u} = \beta_c\mathbf{u}_c + \beta_s\mathbf{u}_s$$

is a general solution to (1.3), and that real initial conditions give rise to real-valued solutions.