Appendix A

Curvilinear coordinates

A.1 Polar coordinates in $\mathbb{R}^2$

We use polar coordinates in $\mathbb{R}^2$ that are related to Cartesian coordinates by

$$x = r\cos\theta \qquad y = r\sin\theta, \tag{A.1}$$

where $r \geq 0$ and $-\pi \leq \theta \leq \pi$. The length element is

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

and the area element is

$$dA = dx\, dy = r\, dr\, d\theta.$$

Exercise A.1. *The chain rule implies that*

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial \theta} &= -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}.
\end{aligned}$$

1. *Use the expressions above to express*

$$\frac{\partial u}{\partial x} \quad \textit{and} \quad \frac{\partial u}{\partial y}$$

in terms of

$$\frac{\partial u}{\partial r} \quad \textit{and} \quad \frac{\partial u}{\partial \theta}.$$

2. *The norm of the gradient of a function u is given by*

$$\| \operatorname{grad} u\|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Use the result of part (1) *of this exercise in order to show that*

$$\| \operatorname{grad} u\|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Exercise A.2. *Use polar coordinates to compute* $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ *by the following trick:*

- *The quantity we want to compute is* $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. *Show that*

$$I^2 = \int_{\mathbb{R}^2} e^{-\|\mathbf{x}\|^2}\,dA.$$

- *Compute* I^2 *by using polar coordinates.*

A.2 Spherical coordinates in $\mathbb{R}^3$

We use spherical coordinates in $\mathbb{R}^3$ that are related to Cartesian coordinates by

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi, \end{aligned} \tag{A.2}$$

$r \geq 0$, $-\pi \leq \theta \leq \pi$, and $0 \leq \phi \leq \pi$. The length element is

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 \sin^2 \phi \, d\theta^2 + r^2 d\phi^2$$

and the area element is

$$dV = r^2 \, \sin \phi \, dr \, d\theta \, d\phi.$$

Exercise A.3. *The 2-dimensional unit sphere (which is given the symbol S^2) is the surface inside $\mathbb{R}^3$ given by $r = 1$. Find the length element and area element of S^2 in coordinates θ, ϕ.*

Exercise A.4. *We now repeat A.1 for spherical coordinates in $\mathbb{R}^3$.*

1. *Use the chain rule in order to express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial x}$ in terms of $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$ and $\frac{\partial u}{\partial \phi}$. If you wish, you may express your result using a matrix.*

2. *Show that*

$$\| \operatorname{grad} u \|^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2 \sin^2 \phi} \left(\frac{\partial u}{\partial \theta} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \phi} \right)^2 .$$