Exercise 1.1 (revised version)

$$M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

$$0 = \begin{vmatrix} 0-\lambda & 1 \\ -\omega^2 & 0-\lambda \end{vmatrix} = (0-\lambda)^2 - (-\omega^2) = \lambda^2 + \omega^2$$

Thus $\lambda^2 = -\omega^2$ and $\lambda = \pm i\omega$

<u>For $\lambda = i\omega$:</u>

$$\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = i\omega \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{array}{l} y = i\omega x \\ -\omega^2 x = i\omega y \end{array} \quad \checkmark$$

choose $x = 1$ so eigenvector is $\begin{pmatrix} 1 \\ i\omega \end{pmatrix}$

<u>For $\lambda = -i\omega$:</u>

$$\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -i\omega \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{array}{l} y = -i\omega x \\ -\omega^2 x = -i\omega y \end{array} \quad \checkmark$$

choose $x = 1$ so eigenvector is $\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$

Since $\frac{dA}{dt} = \lambda A$ we can chose $A = e^{\lambda t}$

Thus scaling solutions are

$$\vec{u}_1 = e^{i\omega t}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} \qquad \vec{u}_2 = e^{-i\omega t}\begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$$

$\boxed{\text{1.2 (revised version)}}$

(1) $$\alpha_1 \begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

becomes system

$$\alpha_1 + \alpha_2 = u_0$$
$$i\omega(\alpha_1 - \alpha_2) = v_0$$

$$\alpha_1 + \alpha_2 = u_0$$
$$\alpha_1 - \alpha_2 = \frac{v_0}{i\omega}$$

Add to get $\quad 2\alpha_1 = u_0 + \frac{v_0}{i\omega}$

Subtract $\quad 2\alpha_2 = u_0 - \frac{v_0}{i\omega}$

Thus

$$\alpha_1 = \frac{1}{2}\left(u_0 - i\,\frac{v_0}{\omega}\right)$$

$$\alpha_2 = \frac{1}{2}\left(u_0 + i\,\frac{v_0}{\omega}\right)$$

(2) Let $\vec{u}(t) = \alpha_1 \vec{u}_1(t) + \alpha_2 \vec{u}_2(t)$

Then $\vec{u}(0) = \alpha_1 \begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$

By part (1) we can choose α_1 and α_2 so that $\vec{u}(0)$ agrees with any $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$. Hence IVP has a solution for any initial condition.

$\boxed{1.3 \text{ Revised version}}$

(1) We have $u_0 = 1$, $v_0 = 0$

Thus $\alpha_1 = \frac{1}{2}$ $\alpha_2 = \frac{1}{2}$ and solution is

$$\vec{u}_c(t) = \frac{1}{2} e^{i\omega t} \begin{pmatrix} 1 \\ i\omega \end{pmatrix} + \frac{1}{2} e^{-i\omega t} \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{i\omega t} + e^{-i\omega t}}{2} \\ \frac{e^{i\omega t} - e^{-i\omega t}}{2} (i\omega) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}$$

(2) $u_0 = 0$, $v_0 = 1$

Thus $\alpha_1 = \frac{-i}{2\omega} = \frac{1}{2\omega i}$

$$\alpha_2 = \frac{i}{2\omega} = \frac{-1}{2\omega i}$$

and

$$\vec{u}_s(t) = \frac{1}{2\omega i} e^{i\omega t}\begin{pmatrix}1\\ i\omega\end{pmatrix} + \frac{-1}{2\omega i} e^{-i\omega t}\begin{pmatrix}1\\ -i\omega\end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{i\omega t}-e^{-i\omega t}}{2\omega i} \\ \frac{e^{i\omega t}+e^{-i\omega t}}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sin(\omega t)}{\omega}\\ \cos(\omega t)\end{pmatrix}$$

(3) Since any real $\begin{pmatrix}u_0\\ v_0\end{pmatrix}$ can be written

as $u_0\begin{pmatrix}1\\ 0\end{pmatrix} + v_0\begin{pmatrix}0\\ 1\end{pmatrix}$ we see that

real initial conditions lead to real solutions.