MATH 131: WORKSHEET 11

Mean value theorems

In this worksheet, and subsequent lectures, we explore results connected to the idea of the *mean (average)* of a function. See §4.4 and §2.10 in the book.

1. The mean value of a function

Suppose we have a function $f(x)$ defined on the interval $a \leq x \leq b$. We define the *mean value* of f, which we give the symbol $\bar{f}$, on that interval to be

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{11.1}$$

Note that the book uses the symbol f_{ave} instead of the symbol $\bar{f}$.

(1) What is the usual notion of the mean/average of a list of numbers? How does the definition of $\bar{f}$ compare? That is, why is this a reasonable definition? Your answer should involve the phrases "cumulative effect" and "size of the interval."

(2) Consider the example of $f(x) = x^3$ on the interval $1 \leq x \leq 4$. What is $f_{\max}$, the largest that f gets? What $f_{\min}$, is the smallest that f gets? What is $\bar{f}$? How do $f_{\max}$, $f_{\min}$, and $\bar{f}$ relate?

(3) Still consider the example $f(x) = x^3$ on the interval $1 \leq x \leq 4$. Show

$$\int_1^4 \bar{f}\,dx = \int_1^4 f(x)\,dx.$$

Why does this make sense? Draw a picture of the graph and give a geometric interpretation.

(4) I claim in general that for any function $f(x)$ on domain $a \leq x \leq b$

$$\int_a^b \bar{f}\,dx = \int_a^b f(x)\,dx.$$

Can you show that this is true using symbolic manipulation? How should one geometrically interpret this?

2. Average rate of change

Suppose again that we have some function $f(x)$ defined on interval $a \leq x \leq b$. We define the *average rate of change on the interval* $a \leq x \leq b$ to be

$$\bar{m} = \frac{f(b) - f(a)}{b - a}.$$

(1) Explain why this is a reasonable definition.

(2) Consider the example of $f(x) = x^3$ on the interval $1 \leq x \leq 4$. What is the average rate of change of f on this interval?

(3) Still with $f(x) = x^3$, let's compare $\bar{m}$ to the derivative of f. Where is $f'(x)$ the largest? Where is it the smallest? How do these largest/smallest values compare to $\bar{m}$?

(4) Draw a picture of the graph of $f(x) = x^3$ on the interval $1 \leq x \leq 4$. Interpret geometrically the largest, smallest, and average rates of change of the function.

(5) In general, how should one interpret geometrically (in terms of the graph) the average rate of change of f.

3. Mean Value Theorems

We give two Mean Value Theorems. Both of them indicate that "at some point the function is average."
Suppose that $f(x)$ is a differentiable function on interval $a \leq x \leq b$.

(1) There exists some number x_* in the interval such that $f(x_*) = \bar{f}$. This means that there is some number such that

$$f(x_*) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

(2) There is some number x'_* such that $\bar{m} = f'(x'_*)$. This means that there is some number such that

$$f'(x'_*) = \frac{f(b) - f(a)}{b - a}.$$

In class, we discuss why these are true... and the relationship between the two of them. For now, I encourage you to find the numbers x_* and x'_* for the function $f(x) = x^3$ on the interval $1 \leq x \leq 4$.