Math 131: Practice for the Midterm Exam – Solutions

1 Computing derivatives using the definition

Compute the derivative of the following functions **using the definition (i.e. limits)**:

1. $f(x) = 2x^2 + 3$

   Solution.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\left[\frac{2(x+h)^2 + 3 - (2x^2+3)}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{4xh + 2h^2}{h}\right] \\
&= \lim_{h\to 0}\,[4x + 2h] \\
&= 4x
\end{aligned}
$$

□

2. $g(x) = \dfrac{1}{x}$

   Solution.

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)\right] \\
&= \lim_{h\to 0}\left[\frac{1}{h}\,\frac{-h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\left[\frac{-1}{x(x+h)}\right] \\
&= \frac{-1}{x^2}
\end{aligned}
$$

□

3. $p(x) = \sqrt{x}$

Solution.

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}\right] \\ &= \lim_{h\to 0}\left[\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right] \\ &= \lim_{h\to 0}\left[\frac{1}{\sqrt{x+h}+\sqrt{x}}\right] \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

□

2 Computing derivatives using shortcut rules

Compute the following derivatives using shortcut rules

1. $f(x) = (x^2+4)^{100}$
2. $g(x) = x\sqrt{x^1+1}$
3. $p(x) = \dfrac{x+3}{x^2+9}$
4. $r(x) = \dfrac{\sqrt{x^2+1}}{x}$

Solution.

1. $$f'(x) = 100(x^2+4)^{99}(2x) = 200x(x^2+4)^{99}$$

2. $$\begin{aligned} g'(x) &= \frac{d}{dx}[x]\sqrt{x+1} + x\frac{d}{dx}\left[(x+1)^{1/2}\right] \\ &= \sqrt{x+1} + x\frac{1}{2}(x+1)^{-1/2} \\ &= \sqrt{x+1} - \frac{x}{2\sqrt{x+1}} \end{aligned}$$

3. $$\begin{aligned} p'(x) &= \frac{d}{dx}\left[(x+3)(x^2+9)^{-1}\right] \\ &= (1)(x^2+9)^{-1} + (x+3)(-1)(x^2+9)^{-2}(2x) \\ &= \frac{1}{x^2+9} - \frac{2x^2+6x}{(x^2+9)^2} \end{aligned}$$

4.
$$\begin{aligned} r'(x) &= \frac{d}{dx}\left[(x^2+1)^{1/2}x^{-1}\right] \\ &= \frac{d}{dx}\left[(x^2+1)^{1/2}\right]x^{-1} + (x^2+1)^{1/2}\frac{d}{dx}\left[x^{-1}\right] \\ &= \frac{1}{2}(x^2+1)^{-1/2}(2x)x^{-1} + \sqrt{x^2+1}\,(-1)x^{-2} \\ &= \frac{1}{\sqrt{x^2+1}} - \frac{\sqrt{x^2+1}}{x^2}. \end{aligned}$$

□

3 Using calculus to understand graphs

For each of the following functions you should

- find the roots, vertical asymptotes, horizontal asymptotes,
- find critical points, regions where the function is increasing/decreasing,
- find infection points, regions where the function is concave up/down
- find local min/max points

Finally, sketch a graph of the function.

1. $f(x) = \dfrac{x}{4+x^2}$

   Solution.

   - Root at $x = 0$; no vertical asymptotes. For horizontal asymptotes we compute
   $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left[\frac{\frac{1}{x}}{\frac{4}{x^2}+1}\right] = 0$$
   Thus the horizontal asymptotes are at $y = 0$.
   - We compute
   $$f'(x) = \frac{4-x^2}{(4+x^2)^2}.$$
   Thus we have critical points at $x = -2$ and $x = 2$.
   The function f is increasing when $-2 < x < 2$.
   The function f is decreasing when $x < -2$ and when $x > 2$.

- We compute
$$f''(x) = \frac{2x^3 - 24x}{(4 + x^2)^3}$$
Thus we have inflection points at $x = 0$, $x = -\sqrt{12}$, $x = \sqrt{12}$.
The graph of f is concave up when $-\sqrt{12} < x < 0$ and when $x > \sqrt{12}$.
The graph of f is concave down when $x < -\sqrt{12}$ and when $0 < x < \sqrt{12}$.
- We have a local minimum at $x = -2$ and a local maximum at $x = 2$.
- The plot of f can be checked on the Desmos calculator online.

□

2. $g(x) = \dfrac{1}{\sqrt{1 + x^2}}$
No solution provided for this one!
3. $p(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$

Solution.
- p has a root at $x = 1$, a vertical asymptote at $x = 0$, and a horizontal asymptote at $y = 0$.
We analyze the vertical asymptote using $p(x) = \frac{x-1}{x^2}$ and thus computing
$$\lim_{x\to 0^+} p(x) = -\infty \quad \text{and} \quad \lim_{x\to 0^-} p(x) = -\infty$$
- We compute
$$p'(x) = \frac{2 - x}{x^3}.$$
We have a critical point at $x = 2$.
The function p is increasing when $0 < x < 2$
The function p is decreasing when $x < 0$ and when $x > 2$
- We compute
$$p''(x) = \frac{2x - 6}{x^4}.$$
We have an inflection point at $x = 3$
The graph of p is concave up when $x > 3$ and is concave down when $x < 3$.
- We have a local maximum at $x = 2$.
- You can check your graph on the Desmos!

□