Math 305 – Midterm Essay

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March 2016

1 Introduction

In this essay we analyze the initial boundary value problem for the one-dimensional wave equation with Dirichlet boundary conditions. We begin our analysis by explaining how the problem can interpreted as describing an oscillating string with fixed endpoints. Once we have motivated the equations comprising the initial boundary value problem, we seek standing wave solutions. We show the process of seeing standing wave solutions yields a Sturm-Liouville eigenvalue problem. In fact, we are able to obtain exact formulas for the eigenfunctions and eigenvalues, and thus obtain exact expressions for the corresonding standing wave solutions. We furthermore show that the energy of each standing wave solution is related to the corresponding eigenvalue, and hence to the frequency at which it oscillates, with higher frequency waves having higher energy. Finally, we use the orthogonality property of the Dirichlet eigenfunctions in order to solve the initial boundary value problem for a particular choice of initial conditions.

2 Motivating the initial boundary value problem

Suppose we want to mathematically describe a string that has fixed endpoints and oscillates according to the wave equation. We can describe the oscillation by a function $u(t, x)$ that gives the horizontal displacement of the string at location x and time t. We require that u satisfy the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}; \tag{1}$$

this can be interpreted as requiring that the acceleration of the displacement is equal to the spatial curvature of the string.

For convenience, we suppose that the string corresponds to x values on the interval $[-1, 1]$. The condition that the endpoints are fixed is mathematically described using the Dirichlet boundary condition

$$u(t, -1) = 0 \quad \text{and} \quad u(t, 1) = 0 \tag{2}$$

for all t.

We can specify that the initial shape of the string correspond to function $s(x)$ by requiring that

$$u(0, x) = s(x). \tag{3}$$

Similarly, we can specify that the initial velocity of each point along the string correspond to the function $v(x)$ by enforcing

$$\frac{\partial u}{\partial t}(0, x) = v(x). \tag{4}$$

If we are given functions $s(x)$ and $v(x)$, then the problem of finding a function $u(t, x)$ satisfying (1), (2), (3), and (4) is called the **initial boundary value problem**.

3 Standing wave solutions

Standing waves are solutions to the wave equation (1) and boundary condition (2) that consist of a spatial shape $\psi(x)$ that is being rescaled in time by amplitude function $A(t)$. Thus to find standing wave solutions, we set $u(t, x) = A(t)\psi(x)$. Inserting this in to (1) and (2) yields

$$\frac{1}{A}\frac{d^2A}{dt^2} = \frac{1}{\psi}\frac{d^2\psi}{dx^2}, \qquad A(t)\psi(-1) = 0, \qquad A(t)\psi(1) = 0.$$

The left side of this first equation is only a function of t, while the right side is only a function of x. Furthermore, we do not want the function $A(t)$ to be identically zero. Therefore, in order to have a nontrivial standing wave solution we must have

$$\frac{d^2A}{dt^2} = \lambda A$$

and

$$\frac{d^2\psi}{dx^2} = \lambda\psi, \qquad \psi(-1) = 0, \qquad \psi(1) = 0 \tag{5}$$

for some constant λ.

The differential equation in (5) takes Sturm-Liouville form

$$\frac{d}{dx}\left[p\frac{d\psi}{dx}\right] + r\psi = \lambda w\psi$$

with $p = 1$, $r = 0$, and $w = 1$. Thus the operator $\frac{d^2}{dx^2}$ is self-adjoint and negative with respect to the usual inner product

$$\langle\psi, \phi\rangle = \int_{-1}^{1} \psi(x)\phi(x)\, dx.$$

Furthermore, if two functions ψ and ϕ both vanish at $x = \pm 1$, then

$$\left[\psi\frac{d\phi}{dx} - \phi\frac{d\psi}{dx}\right]_{-1}^{1} = 0.$$

Thus we see that (5) satisfies the conditions in the Sturm-Liouville theorem. This means that there exists an infinite list of eigenvalues λ_k and corresponding list of eigenfunctions $\psi_k(x)$ such that

$$\frac{d^2\psi_k}{dx^2} = \lambda_k \psi_k, \qquad \psi_k(-1) = 0, \qquad \psi_k(1) = 0$$

and such that the eigenfunctions ψ_k form a complete orthogonal collection.

In fact, we can find explicit formulas for these eigenfunctions and eigenvalues. First we show that the eigenvalues are negative. Multiplying the differential equation in (5) by ψ, integrating, and then using integration by parts yields

$$-\int_{-1}^{1} \left(\frac{d\psi}{dx}\right)^2 dx = \lambda \int_{-1}^{1} \psi(x)^2 \, dx.$$

Thus we see that $\lambda \leq 0$ and we may write $\lambda = -\omega^2$ for some $\omega \geq 0$.

Since $\lambda = -\omega^2$ we want to solve the differential equation

$$\frac{d^2\psi}{dx^2} = -\omega^2 \psi.$$

The general solution is a linear combination of

$$\sin(\omega x) \quad \text{and} \quad \cos(\omega x).$$

Imposing the boundary conditions we see that for each integer l the function $\sin(\omega x)$ solves (5) if $\omega = l\pi$ and the function $\cos(\omega x)$ is a solution if $\omega = (2l+1)\pi/2$. Thus the eigenfunctions solving (5) are

$$\psi_k(x) = \begin{cases} \cos\left(\frac{2l+1}{2}\pi x\right) & \text{if } k \text{ is odd and } k = 2l+1, \\ \sin\left(l\pi x\right) & \text{if } k \text{ is even and } k = 2l. \end{cases}$$

The corresponding eigenvalues are $\lambda_k = -\omega_k^2$, where $\omega_k = \frac{k\pi}{2}$. Notice that ψ_k is an even function when k is odd, and is an odd function when k is even.

In order to complete our construction of standing wave solutions we must, for each positive integer k, find a function $A(t)$ satisfying $A''(t) = -\omega_k^2 A(t)$. This is easily done – solutions are

$$\cos(\omega_k t) \quad \text{and} \quad \sin(\omega_k t).$$

Combining this with our formulas for the eigenfunctions $\psi_k(x)$ we see that for each nonnegative integer l we have the following standing wave solutions:

$$\cos\left(\frac{2l+1}{2}\pi t\right)\cos\left(\frac{2l+1}{2}\pi x\right),$$
$$\sin\left(\frac{2l+1}{2}\pi t\right)\cos\left(\frac{2l+1}{2}\pi x\right),$$
$$\cos(l\pi t)\sin(l\pi x),$$
$$\sin(l\pi t)\sin(l\pi x).$$

We conclude this section by examining the energy of each of these standing waves. We know that for any solution $u(t,x)$ so the wave equation (1) the quantity

$$E = \frac{1}{2}\int_{-1}^{1}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\} dx$$

is constant in time.

By direct computation we find that the energy corresponding to a standing wave solution $A(t)\psi_k(x)$ is

$$E_k = \frac{1}{2}|\lambda_k| = \frac{1}{2}\left(\frac{k\pi}{2}\right)^2.$$

Thus the eigenvalue λ_k of an eigenfunction precisely tells us the energy of the corresponding standing wave solutions. We notice that the standing wave solutions corresponding to ψ_k oscillate at frequency

$$\omega_k = |\lambda_k|^{1/2} = \frac{k\pi}{2}.$$

Thus high-frequency solutions have larger energy. This agrees with notions about energy coming from physics.

4 Solving the initial boundary value problem

We now use the completeness property of the eigenfunctions $\psi_k(x)$ in order to solve the initial boundary value problem with

$$s(x) = 1 - |x| \quad \text{and} \quad v(x) = 0. \tag{6}$$

The linearity of the wave equation implies that the general solution to (1) satisfying the boundary condition (2) is

$$u(t,x) = \sum_{l=0}^{\infty}\left\{a_l\cos\left(\frac{2l+1}{2}\pi t\right)\cos\left(\frac{2l+1}{2}\pi x\right) + b_l\sin\left(\frac{2l+1}{2}\pi t\right)\cos\left(\frac{2l+1}{2}\pi x\right)\right\}$$
$$+ \sum_{l=1}^{\infty}\left\{c_l\cos(l\pi t)\sin(l\pi x) + d_l\sin(l\pi t)\sin(l\pi x)\right\}.$$

Thus in order to satisfy the initial conditions we want

$$1 - |x| = \sum_{l=0}^{\infty} a_l \cos\left(\frac{2l+1}{2}\pi x\right) + \sum_{l=1}^{\infty} c_l \sin(l\pi x) \tag{7}$$

and

$$0 = \sum_{l=0}^{\infty} b_l \left(\frac{2l+1}{2}\pi\right) \cos\left(\frac{2l+1}{2}\pi x\right) + \sum_{l=1}^{\infty} d_l (l\pi) \sin(l\pi x)\,. \tag{8}$$

We can arrange that (8) hold by simply requiring that $b_l = 0$ and $d_l = 0$ for all l.

In order to satisfy (7) we use the completeness and orthogonality of the eigenfunctions, which implies that we may write

$$1 - |x| = \sum_{k=1}^{\infty} \alpha_k \psi_k$$

provided

$$\alpha_k = \frac{\langle 1 - |x|, \psi_k \rangle}{\|\psi_k\|^2}.$$

Since $1 - |x|$ is an even function, and since ψ_k is an odd function when k is even, we have that $\alpha_k = 0$ whenever k is even. Thus with $k = 2l + 1$ we compute

$$\langle 1 - |x|, \psi_k \rangle = 2 \int_0^1 (1 - x) \cos\left(\frac{2l+1}{2}\pi x\right)\, dx = \frac{8}{\pi^2 (2l+1)^2}.$$

Since $\|\psi_k\|^2 = 1$ we have

$$1 - |x| = \sum_{l=0}^{\infty} \frac{8}{\pi^2 (2l+1)^2} \cos\left(\frac{2l+1}{2}\pi x\right).$$

From this we conclude that $c_l = 0$ and that the solution to the initial boundary value problem (1)–(2)–(3)–(4) with initial conditions given by (6) is

$$u(t,x) = \sum_{l=0}^{\infty} \frac{8}{\pi^2 (2l+1)^2} \cos\left(\frac{2l+1}{2}\pi t\right) \cos\left(\frac{2l+1}{2}\pi x\right). \tag{9}$$

An illustration of this function is given in Figure 1.

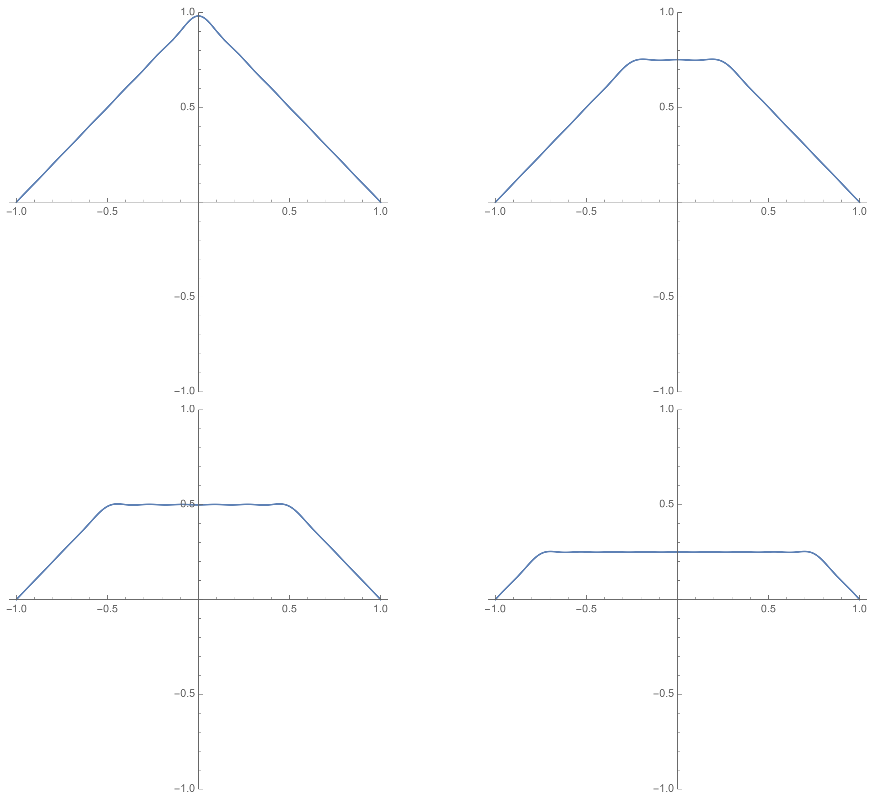

Figure 1: In order to illustrate the function $u(t, x)$ given by (9) we plot the partial sum determined by $0 \leq l \leq 10$ at times $t = 0$ (upper left), $t = 0.25$ (upper right), $t = .50$ (lower left), and $t = .75$ (lower right). The solution satisfies $u(1, x) = 0$, and oscillates periodically, so that $u(2, x) = -u(0, x)$ and $u(4, x) = u(0, x)$.