DAY 1

What is a differential equation?

ifferential-equation

Reading: §1.1 Modeling via differential equations

Differential equations:

- $\dfrac{dQ}{dt} = \dfrac{\text{change in } Q}{\text{change in } t}$
- units
- Important: measures rate of change, *not* value at any given time

Key expressions:

- rate of change / growth rate
- relative rate of change / relative growth rate
- is constant
- is proportional to

Translation:

- Rate of change is constant
- Basic growth model: Relative growth rate is constant
- Important: "5% growth" means *relative growth*
- Logistic model: Relative growth rate is proportional to available habitat

Differential equations vs. algebraic equations

- Type of unknown
- Questions: Existence of solutions? Uniqueness of solutions? Properties?
- Example: $e^{-y} = y$ vs. $\dfrac{dy}{dt} = y$

Systems of differential equations

- Multiple unknowns

- Example

$$\frac{dx}{dt} = x^2 - y^2 \qquad \frac{dy}{dt} = x + y$$

FirstConstruct

Exercise 1.1. Construct a differential equation which models the following situations.

(1) An investment $y(t)$ grows with relative growth rate of 5% per year;
(2) \$1 000 is invested with annual interest rate of 5%;
(3) Continuous deposits are made into an account at the rate of \$1000 a year. Regardless of these additional deposits, the account earns 7% a year.
(4) Paul takes out a loan with an annual interest rate of 6%. Continuous repayments are made totaling \$1 000 per year.
(5) A fish population under ideal conditions grows at a relative growth rate k per year. The carrying capacity of their habitat is N and H fish are harvested each month.
(6) A fish population under ideal conditions grows at growth rate k per year. The carrying capacity of their habitat is N and one quarter of the fish population is harvested annually.
(7) Due to the pollution problems the relative growth rate of a fish population is a decreasing exponential function of time.
(8) The population sizes of two competing species A and B, denoted $x_A(t)$ and $x_B(t)$, grow with relative growth rates which are inversely proportional to the sizes of their competitors' population.

DAY 2

Modeling with differential equations

chapter:ReadAndWrite

Reading: §1.1 Modeling via Differential Equations

Modeling

- What is a model? "Physical" situation vs. mathematical description
- Modeling process: Physical situation $\rightsquigarrow$ idealized situation $\rightsquigarrow$ mathematical description $\rightsquigarrow$ mathematical analysis $\rightsquigarrow$ interpretation
- Asking questions: "physical question" vs. mathematical question

Logistic model

- $\frac{1}{P}\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)$
- Given $K = 1500$ and $r = 5\%$, what is rate of change when $P = 0.5$?
- Plot
- modifications: harvesting (+const, $+\alpha P$, etc.), hatchery (similar)

Classic mixing problem 1

- Tank with 100 gallons fresh water
- Drain at rate of 5 gallons/minute
- Add salt concentrate (8 oz/gal) at rate of 5 gal/min; instant mixing
- How much salt in tank as function of time?
- S = salt (oz)

$$\frac{dS}{dt} = ((\text{rate of) salt in}) - (\text{salt out}) = \cdots = 40 - \frac{1}{20}S \qquad (\text{oz/min})$$

Classic mixing problem 2

- Brother has 22oz chocolate milk, drinking at rate of 3 oz/min
- I'm adding strawberry milk (10% fruit) at rate of 1 oz/min
- How much fruit in glass as a function of time?

Simple physics

- Newton: $F = ma$
- Hooke (SHO): $F = -kx \quad \rightsquigarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$
- Pendulum: $F = -mg\sin\theta \quad \rightsquigarrow \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$
- Simple friction: $F = -\beta\frac{dx}{dt}$

Equations with parameters:

- Logistic: As $K \to \infty$ we obtain basic growth model
- SHO: As $m \to \infty$ we get free motion

Vocabulary:

- First-order equations: Involve first derivatives
- Second-order equations: Involve second derivatives
- System of equations

Modeling1

Exercise 2.1. Some quantity $f = f(t)$ which changes with time and is measured in gallons and is modeled by the differential equation

$$\frac{df}{dt} = k\,f\left(1 - \frac{f}{M}\right).$$

The time variable t is measured in months and the parameter M is also measured in gallons.

(1) In what units is $\frac{df}{dt}$ measured?
(2) In what units is $1 - \frac{f}{M}$ measured?
(3) In what units is $f \cdot \left(1 - \frac{f}{M}\right)$ measured?
(4) In what units should the parameter k be measured in order for the equality to hold?

Modeling2

Exercise 2.2. A variable quantity $r = r(t)$ is measured in grams per liter and is modeled by the differential equation

$$\frac{dr}{dt} = -k \cdot \frac{r}{c+r}.$$

The time variable t is measured in seconds. Figure out the units for the parameters k and c.

RW-MidBegin

Exercise 2.3. We consider a large urn of coffee in The Bon. A student is draining coffee in to her thermos while at the same time the brewing machine is adding fresh coffee. We want to keep track of the amount of caffeine (in milligrams) in the urn during this process.

Suppose the following:

- There is initially 4 liters of coffee in the urn, with a caffeine concentration of 100 mg per liter.
- Starting at time $t = 0$ the brewing machine is adding extra strong coffee, which has a concentration of 250 mg per liter. This new coffee is being added at a rate of 25 mL per minute. At what rate is caffeine entering the urn?
- Starting at time $t = 0$ the student begins draining coffee out of the urn at a rate of 30 mL per minute. Write down an expression for the volume of coffee in the urn as a function of time.

Write down a differential equation which models the amount of caffeine in the urn as a function of time.

`IvaJunkEmail`

Exercise 2.4. In this problem we model the number of junk emails in Iva's inbox with a continuous function $J(t)$. Suppose the following:

- When the semester started ($t = 0$), Iva had 6000 emails in her inbox; 4000 of them were junk.
- Email is continuously flowing in to Iva's email inbox at a rate of 50 per day; 30% of the emails are junk.
- Each day, Iva randomly picks 20 emails to deal with[1] – once they have been dealt with, she moves them out of her inbox.

Write down a differential equation describing the number of junk emails in Iva's inbox.

`RW-Last`

Exercise 2.5. A big mixing vat contains 50 liters of a mixture in which the concentration of a certain chemical is 1.25g/L. This mixture is being diluted by another mixture in which the concentration of the same chemical is 0.25g/L. Each minute 6 liters of the less concentrated mixture are poured into the vat and 4 liters of the resulting new mixture are drained out. Model this process.

`MathPhysicsPrograms`

Exercise 2.6. Due to the high quality of the faculty, the physics and mathematics programs at Lois & Claire College would – under ideal circumstances – both grow at a relative growth rate of 250%. However, due to the limited number of faculty advisors, there is a maximum of 75 physics majors and 175 math majors.

(1) Write down logistic growth models for each of the two populations.

[1] This is not actually true, of course.

(2) In reality, students actually graduate. Modify your differential equations to take in to account the fact that each year roughly a third of the majors in each department leave due to graduation.

(3) Now suppose (counterfactually) that the two departments are in competition, meaning that there is negative contribution to the growth rate of physics majors of the form

$$-5(\text{number of physics majors})(\text{number of math majors})$$

and a similar contribution to the growth rate of math majors of the form

$$-3(\text{number of physics majors})(\text{number of math majors})$$

Write down the resulting system of differential equations describing the number of math and physics majors.

SoccerFootball

Exercise 2.7. According to reputable internet sources, a typical soccer (football) ball has a mass of 0.425 kg. Near the surface of the earth, the force of gravity is $F = -mg$, where g is approximately $9.8\ \mathrm{m/s^2}$ and the minus sign indicated downward force.

(1) Write a differential equation describing the height of the ball as a function of time.

(2) Integrate twice to find the most general solution to the differential equation.

w-long-is-ball-aloft

(3) Suppose the ball is kicked from a height of 1 m with an initial upwards velocity of $10\ \mathrm{m/s}$. How long will the ball remain in the air?

(4) Reflect on your computation in part 3. How does the mass of the ball affect your answer? (For example, how would the answer change if the ball was much heavier or much lighter?)