More sophisticated viewpoint

- *Propagator function* $P(t) = \exp\left(\int_0^t r(\tau)\,d\tau\right)$
- $y(t) = P(t)\,y_0$ means take value y_0 and propagate it forward according to ODE by time t
- What if we wanted to start at time $t = s$ instead of time $t = 0$? Use
$$\exp\left(\int_s^t r(\tau)\,d\tau\right) = \exp\left(\int_0^t r(\tau)\,d\tau - \int_0^s r(\tau)\,d\tau\right)$$
$$= \frac{P(t)}{P(s)}$$
- Properties of the propagator function (HW)
$$P(0) = 1, \qquad P'(t) = r(t)P(t).$$

How to include forcing?

- Viewpoint: At each time s we have additional input $f(s)$... which needs to be propagated forward from time s to time t.
- Contribution of forcing is
$$\int_0^t P(t)\frac{1}{P(s)}f(s)\,ds = P(t)\int_0^t \frac{1}{P(s)}f(s)\,ds$$
- If IVP is
$$\frac{dy}{dt} = r(t)y + f(t) \qquad y(0) = y_0$$
solution is
$$y(t) = \underbrace{P(t)\,y_0}_{\text{due to } y_0} + \underbrace{P(t)\int_0^t \frac{1}{P(s)}f(s)\,ds}_{\text{due to forcing}}.$$
This formula is called *Duhamel's formula*
- Emphasize structure – two terms: one due to initial condition, one due to forcing

Example: $\frac{dy}{dt} = t^2 y$, $y(0) = 17$

- $r(t) = t^2$, $f(t) = 0$
- $P(t) = e^{t^3/3}$
- $y(t) = 17e^{t^3/3}$

Example: $\frac{dy}{dt} = \cos t - 7y$, $y(0) = \sqrt{17}$

- $r(t) = -7$, $f(t) = \cos t$
- $P(t) = e^{-7t}$
- $y(t) = e^{-7t}\sqrt{17} + e^{-7t} \displaystyle\int_0^t e^{7s} \cos s \, ds = \dots$

Example: $\frac{dy}{dt} = 1 - \frac{y}{1-t}$, $y(0) = 31$

- Note: equation is only valid for $t < 1$ (since we start at $t = 0$)
- $r(t) = -\frac{1}{1-t}$, $f(t) = 1$
- $P(t) = 1 - t$
- $y(t) = 31(1-t) + (1-t)\int_0^t \frac{1}{1-s} ds = \dots$

Exercise 7.1. Verify (by direct computation) the three properties of the propagator function $P(t)$:

(1) $P'(t) = r(t)P(t)$,
(2) $P(0) = 1$.

Exercise 7.2. Solve the IVP:

$$\frac{dy}{dt} = 2y - t \qquad y(0) = 1.$$

Exercise 7.3. Solve the IVP:

$$\frac{dy}{dt} = \frac{y}{1+2t} - 7 \qquad y(0) = 1.$$

Exercise 7.4. Suppose money is invested in a volatile market that has an annual growth rate of $r(t) = 0.01 + 0.05 \cos 10t$, where t is measured in years.

(1) Make a plot of $r(t)$ over a 10 year time period. How should one interpret this growth rate?
(2) Suppose there is an initial investment of \$100. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?
(3) Suppose instead that no money is initially invested, but that one continuously adds to an investment at a rate of \$10 per year. Make a plot of the value of the investment over a 10 year time period? What is the value at the end of the 10 years?

Exercise 7.5. Solve the initial value problem from Exercise 2.3.