Instructions:

- Complete all problems on the paper I provide; write on only one side of each page.
- You must show legible work in order to receive credit.
- If you are not confident in some result, you will receive more credit if you make a note of this and comment on where you might be going wrong.
- Use of technology (calculator, cell phone, etc.) is not permitted.

Problem 1. A population of coddling moths lives in an apple orchard. In this problem you model the population under a variety of assumptions. Let P represent the number of moths, measured in thousands, and measure time t in years.

A. Assume that the population increases with a relative growth rate of 500% per year. What is the corresponding model for the population?

B. Suppose in addition that the orchard can only sustainably support 10,000 moths. Write down the corresponding logistic model. What are the assumptions that the logistic model describes?

C. Suppose that the owners of the orchard employ pheromone traps that capture moths at a rate of 8,000 per year. Write down a modified logistic model that describes this situation.

D. Find the equilibrium solutions to the modified logistic model from part C.

E. (Bonus–attempt only after completing the rest of the exam) Sketch the slope field for the modified logistic model you found in part C. What do you predict if there are initially 3000 moths in the orchard? What if there are initially 1000 moths?

Solution.

A. The model is $\frac{dP}{dt} = 5P$.

B. The model is $\frac{dP}{dt} = 5P\left(1 - \frac{P}{10}\right)$.
The assumption behind the model is that the relative growth rate is proportional to the percent of available habitat.

C. The model is $\frac{dP}{dt} = 5P\left(1 - \frac{P}{10}\right) - 8$.

D. The equilibrium solutions are found by solving

$$0 = 5P\left(1 - \frac{P}{10}\right) - 8.$$

After some algebra, we find that the equilibria are $P = 2$ *and* $P = 8$.

Problem 2. Consider the following initial value problem:

$$\frac{dS}{dt} = (20 \text{ liters per minute}) \left(e^{-2t} \text{ grams of salt per liter}\right) - (20 \text{ liters per minute}) \left(\frac{S \text{ grams of salt}}{100 \text{ liters}}\right)$$

$$S(0) = 7 \text{ grams of salt.}$$

A. Describe in words the physical situation that the IVP models.

B. Solve the initial value problem.

Solution.

This IVP describes a mixing problem that involves a 100 *liter tank. Initially there are* 7 *grams of salt in the tank. A salt solution is flowing in to the tank at* 20 *liters per minute; mixed solution is flowing out at that same rate. The concentration of the ingoing solution changes with time—it contains* e^{-2t} *grams of salt per liter.*

To solve the IVP we write the ODE as

$$\frac{dS}{dt} = \underbrace{-\frac{1}{5}}_{r(t)} S + \underbrace{20e^{-2t}}_{f(t)}.$$

The propagator function is given by

$$P(t) = \exp\left(\int_0^t -\frac{1}{5}\, d\tau\right) = e^{-t/5}.$$

Thus the solution to the IVP is

$$\begin{aligned}
S(t) &= P(t)S_0 + P(t)\int_0^t \frac{1}{P(s)} f(s)\, ds \\
&= 7e^{-t/5} + e^{-t/5}\int_0^t e^{s/5} 20e^{-2s}\, ds \\
&= 7e^{-t/5} + e^{-t/5}\left[20\left(-\frac{5}{9}\right) e^{-\frac{9}{5}s}\right]_0^t \\
&= \frac{163}{9} e^{-t/5} - \frac{100}{9} e^{-2t}
\end{aligned}$$