In-class activity regarding $\mathbb{P}_3$

Here we study the linear transformation $H\colon \mathbb{P}_3 \to \mathbb{P}_3$ given by

$$\frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

(1) Verify that H is indeed a linear transformation $\mathbb{P}_3 \to \mathbb{P}_3$
(2) Find a matrix $M_{\mathcal{E}}$ which expresses H relative to the standard basis $\mathcal{E}$.
(3) Find a basis for the null space of $M_{\mathcal{E}}$. Use this to find a basis for $\ker(H)$.
(4) Find a basis for the columnspace of $M_{\mathcal{E}}$. Use this to find a basis for $\operatorname{im}(H)$.
(5) Verify that the Rank-Nullity theorem holds for the transformation H.
(6) Let $q = 1 + 2x + 3x^2 + 4x^3$. Describe the space of all solutions p to the equation $H(p) = q$.
(7) Let $r = 1 + 2x - 3x^2 - 4x^3$. Describe the space of all solutions p to the equation $H(p) = r$.
(8) Find the eigenvalues and eigenspaces of $M_{\mathcal{E}}$. Use these to determine the eigenvalues and eigenspaces of H.
(9) Construct a basis for $\mathbb{P}_3$ consisting of eigen-polynomials p of H which have the property that $p(1) = 1$. Call this basis $\mathcal{B}$.
(10) Construct a matrix S which converts between the basis $\mathcal{E}$ to basis $\mathcal{B}$.
(11) Use the matrix S to convert the matrix $M_{\mathcal{E}}$ to the matrix $M_{\mathcal{B}}$ which describes H relative to coordinates determined by $\mathcal{B}$.