Math 225 **Exam 1 – Practice** Fall 2015

Here are some problems to help you practice for the first exam.

Problem 1. Give careful definitions of the following terms. (Be sure to give the "formal" definition and not a preliminary "intuitive" definition.)

A. A *subspace* of $\mathbb{R}^n$.

B. A *linear transformation*.

C. The *kernel and range* of a linear transformation.

D. A *linearly independent* collection of vectors.

E. The *span* of a collection of vectors.

F. A *basis* for a subspace.

G. A *consistent* system of equations.

Solution.

A. A subspace is a collection of vectors having the properties that (i) if two vectors are in the collection then so is their sum and (ii) if a vector is in the collection then so is any rescaling of that vector.

B. A linear transformation is a function that respects addition and scaling in the sense that

$$f(\mathbf{v}+\mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w}) \qquad f(\alpha\mathbf{v}) = \alpha f(\mathbf{v})$$

for any vectors $\mathbf{v}, \mathbf{w}$ *and number* α*.*

C. The kernel of a linear transformation is the subspace of the domain containing all vectors that are mapped to zero. The range of a linear transformation is the collection of all vectors in the codomain that are outputs of the transformation.

D. A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *is linearly independent if the only solution to*

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = 0$$

is $\alpha_1 = 0,\ \ldots,\ \alpha_k = 0$.

E. The span of a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *is the collection of all vectors of the form*

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$$

where $\alpha_1, \ldots, \alpha_k$ *are numbers.*

F. A collection of vectors forms a basis of a subspace if (i) the collection is linearly independent and (ii) the span of the collection is the entire subspace.

G. A system of equations is consistent if there exists a solution to the system.

Problem 2.

A. State the Rank-Nullity Theorem.

B. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^4$ has a one dimensional kernel. What else can you say about f?

C. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^2$. What can you say about the kernel of f?

D. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ has kernel equal to $\{\mathbf{0}\}$. Discuss the solvability of $f(\mathbf{v}) = \mathbf{r}$.

E. Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ has $\operatorname{ran}(f) = \mathbb{R}^2$. Discuss the solvability of $f(\mathbf{v}) = \mathbf{r}$.

F. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Under what conditions is $f(\mathbf{v}) = \mathbf{0}$ solvable?

Solution.

A. The rank nullity theorem says that if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$\dim(\ker(f)) + \dim(\operatorname{ran}(f) = n.$$

B. The dimension of the range must be two.

C. The kernel must have dimension greater or equal to one.

D. Since the kernel is trivial, the dimension of the range is three. Thus the range is all of $\mathbb{R}^3$ and the equation has a solution for every $\mathbf{r}$. Since the kernel has only one element, the solution is always unique.

E. Since the range is $\mathbb{R}^2$, there is always a solution. Since the dimension of the kernel is one, there is always a line of solutions.

F. The equation is always solvable because the zero vector is always a solution.

Problem 3. Consider the system of equations

$$\begin{aligned} 12x - 9y + 3z - 2w &= 57 \\ -12x + 9y + 3z + 4w &= -63 \\ 8x - 6y - 3z - 3w &= 43 \\ -8x + 6y - 3z + w &= -37 \end{aligned}$$

(Note: The numbers here aren't the best... I'm saving the better numbers for the actual exam!)

A. Find the space of solutions to this system. Describe the solution space geometrically.

B. Express the system of equations in terms of a linear transformation f. State the domain and codomain of f.

C. What is the kernel of f? Describe using a basis, and also give a geometric description.

D. What is the range of f? Describe using a basis and also give a geometric description.

Solution.

A. We put the numbers in to a bit augmented matrix and row reduce

$$\left(\begin{array}{cccc|c} 12 & -9 & 3 & -2 & 57 \\ -12 & 9 & 3 & 4 & -63 \\ 8 & -6 & -3 & -3 & 43 \\ -8 & 6 & -3 & 1 & -37 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & -3/4 & 0 & -1/4 & 5 \\ 0 & 0 & 1 & 1/3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Thus the space of solution is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3/4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1/4 \\ 0 \\ -1/3 \\ 1 \end{pmatrix}$$

Geometrically, the space of solution is a plane.

B. We can express the system of equations as

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 57 \\ -63 \\ 43 \\ -37 \end{pmatrix}, \qquad \text{where} \qquad f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12x - 9y + 3z - 2w \\ -12x + 9y + 3z + 4w \\ 8x - 6y - 3z - 3w \\ -8x + 6y - 3z + w \end{pmatrix}$$

Both the domain and codomain of f are $\mathbb{R}^4$

C. Using the row reducing above (with zero on the right side) we see that

$$\ker(f) = \text{span}\left\{ \begin{pmatrix} 3/4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 0 \\ -1/3 \\ 1 \end{pmatrix} \right\}$$

This is a two-dimensional subspace, and is thus a plane.

D. We have

$$\operatorname{ran}(f) = \operatorname{span}\left\{ \begin{pmatrix} 12 \\ -12 \\ 8 \\ -8 \end{pmatrix}, \begin{pmatrix} -9 \\ 9 \\ -6 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \\ 1 \end{pmatrix} \right\},$$

but these vectors are not linearly independent.

Using the row reducing above (again, with zero on the right side) we see that the equation

$$\alpha_1 \begin{pmatrix} 12 \\ -12 \\ 8 \\ -8 \end{pmatrix} + \alpha_2 \begin{pmatrix} -9 \\ 9 \\ -6 \\ 6 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 3 \\ -3 \\ -3 \end{pmatrix} + \alpha_4 \begin{pmatrix} -2 \\ 4 \\ -3 \\ 1 \end{pmatrix} = \mathbf{0}$$

has solutions

$$\begin{aligned} \alpha_1 &= \frac{3}{4}\alpha_2 + \frac{1}{4}\alpha_4 \\ \alpha_2 &= \textit{anything} \\ \alpha_3 &= -\frac{1}{3}\alpha_4 \\ \alpha_4 &= \textit{anything} \end{aligned}$$

This means that the second and fourth vectors can be built from the first and third. Thus

$$\operatorname{ran}(f) = \operatorname{span}\left\{ \begin{pmatrix} 12 \\ -12 \\ 8 \\ -8 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \\ -3 \end{pmatrix}, \right\}$$

and the range is a two-dimensional subspace of $\mathbb{R}^4$.

Problem 4. Consider the transformation $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 6x+2y \end{pmatrix}$

A. What is the determinant of this transformation? Give a geometric interpretation.

B. Find the eigenvalues of this transformation. For each eigenvalue, find the corresponding eigenspace.

C. Explain how we know that f has an inverse transformation. Find the formula for f^{-1}.

Solution.

A. The determinant is $\det(f) = -10$. *This means that the transformation deforms area by a factor of* 10 *and that there is a reversal/flip done by the transformation.*

B. The eigenvalues are those for values λ *for which* $\det(f - \lambda\,\mathrm{id}) = 0$, *and thus must satisfy* $0 = (1-\lambda)(2-\lambda) - 12$. *The solutions are* $\lambda = 5$ *and* $\lambda = -2$.

For $\lambda = 5$ *the eigenspace is the kernel of* $f - 5\,\mathrm{id}$. *This can be computed by row reducing*

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 6 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus the eigenspace is the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda = -2$ *the eigenspace is the kernel of* $f + 2\,\mathrm{id}$. *This can be computed by row reducing*

$$\left(\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & 4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus the eigenspace is the span of $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

C. Since the determinant of the linear transformation is not zero we know that the kernel is zero and that the transformation is invertible. We compute the inverse by row reducing

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{10} \end{array}\right)$$

Thus

$$f^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}x + \frac{1}{5}y \\ \frac{3}{5}x - \frac{1}{10}y \end{pmatrix}$$