TOPIC 19

Review for the final exam

Exercise 19.1. Understand how to solve linear systems of equations.

(1) Explain how to use matrix notation to express linear systems.
(2) Define the following terms: homogeneous system, inhomogeneous system, consistent system, inconsistent system, underdetermined system, overdetermined system.
(3) Explain what *row reduction of an augmented matrix* is, and why it is mathematically legitimate. (That is, what – mathematically – are we doing when we perform row reduction?)
(4) Explain why all homogeneous systems are consistent?
(5) Find all solutions to the following linear systems:

Exercise 19.2. Understand the geometry behind linear systems.

(1) How can we use linear systems of equations to describe lines, planes, etc.?
(2) Revisit some of the earlier homework problems involving lines and planes.

Exercise 19.3. Understand the vector space $\mathbb{R}^n$?

(1) What is the vector space $\mathbb{R}^n$?
(2) Define the following terms: linearly independent, linearly dependent, subspace (of $\mathbb{R}^n$), span, basis,
(3) Explain how to check whether a collection of vectors is linearly dependent or linear dependent.
(4) Consider the vectors

$$\begin{pmatrix}1\\2\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Are these vectors linearly independent? Describe the subspace that is the span of these vectors

(5) Revisit earlier homework problems in which you had to find a basis for certain subspaces.

Exercise 19.4. Understand how to work with matrices.

(1) What is a matrix? How to we add/multiply/scale matrices?
(2) Give definitions of the following terms: square matrix, symmetric matrix, identity matrix, inverse of a matrix, transpose of a matrix, determinant of a matrix, singular matrix, upper- (lower-)triangular matrix, diagonal matrix
(3) What are the elementary matrices? Explain their importance.
(4) Explain, geometrically, what the determinant of a matrix represents.
(5) Explain how to compute the determinant of a matrix.
(6) Revisit earlier homework problems in which you had to (a) determine whether a matrix was invertible or not and (b), in the case that it was, compute the inverse matrix.

Exercise 19.5. Understand the relationship between determinants, invertibility of matrices, and solving linear systems.

(1) Explain the relationship between the determinant of a matrix and the linear indpendence of its column (or row) vectors.
(2) Revisit earlier homework problems in which you have to decide whether a given collection of vectors is linearly independent or not.
(3) Explain the relationship between the determinant of matrix A and the solvability of the system $A\mathbf{x} = \mathbf{b}$. What are the three options regarding the solvability of such a system? When does each option occur?
(4) Construct several linear systems and discuss their solvability.

Exercise 19.6. Understand linear transformations of $\mathbb{R}^n \to \mathbb{R}^m$

(1) What is a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$? What does it mean to say that a linear transformation is given by matrix A?
(2) Define the following words: kernel, image,
(3) Pick several linear transformations and describe geometrically what they do.
(4) What is the rank-nullity theorem for linear transformations $\mathbb{R}^n \to \mathbb{R}^m$?
(5) Pick several "interesting" linear transformations. For each find the kernel, the image, and discuss the rank-nullity theorem.
(6) In class we discussed 'two worlds' describing two different possibilities for a transformation $\mathbb{R}^n \to \mathbb{R}^n$. What are these two possibilities, and what are the long list of equivalent conditions in each case?

Exercise 19.7. Understand eigenstuff.

(1) What is the eigenvalue problem? Your answer should include both an algebraic part and a geometric part.
(2) Give definitions of the terms: eigenvalue, eigenvector, eigenspace, characteristic polynomial.

(3) Find the eigenvalues and eigenspaces of several transformations $\mathbb{R}^n \to \mathbb{R}^n$. (You need to be able to compute by hand!)
(4) Suppose $\lambda = 0$ is an eigenvalue of transformation T. Is T invertible? Explain.
(5) Do all transformations have (real) eigenvalues? Explain.

Exercise 19.8. What is a vector space? Give the definition and three examples.

(1) Give definitions of the following in the general vector space setting:
  - subspace
  - linear (in)dependence, span, basis, dimension
  - linear transformation, kernel, image, invertibility,
  - eigenvalue, eigenspace, eigenvector

(2) Explain why the collection of all continuous functions with domain $[-1, 1]$ forms a vector space. Call this vector space $C([-1, 1])$. Explain how the sub-collection of all function $f(x)$ satisfying
$$\int_{-1}^{1} f(x)\,dx = 0$$
is a subspace of $C([-1, 1])$.
(3) What is the vector space $M_{2\times 2}$? Explain how the collection of anti-symmetric matrices is a subspace. Find a basis for $M_{2\times 2}$ and a basis for the subspace of anti-symmetric matrices.
(4) What is the vector space $\mathbb{P}_n$? Explain why the collection of even polynomials is a subspace. Find a basis for both $\mathbb{P}_n$ and also a basis for the subspace of even polynomials.
(5) Give an example of a linear transformation $M_{2\times 2} \to M_{2\times 2}$.
(6) Give an example of a linear transformation $\mathbb{P}_{10} \to \mathbb{P}_{11}$.

Exercise 19.9. Understand how to use coordinates to study a vector space.

(1) Suppose V is a vector space with basis $\mathcal{B}$. Explain what it means to express v (and element of V) in coordinates relative to $\mathcal{B}$.
  - What is the 'standard' basis $\mathcal{B}$ for $\mathbb{P}_3$? Express the polynomial $1 - x + x^2 - x^3$ in coordinates relative to this basis.
  - What is the 'standard' basis for $M_{2\times 2}$? Express the matrix
  $$M = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$$
  in coordinates relative to this basis.

(2) Suppose $T\colon V \to W$ is a linear transformation, and that $\mathcal{B}$ is a basis for V and C is a basis for W. What does it mean for matrix M to represent T in coordinates relative to these bases?

- Suppose $T\colon \mathbb{P}_3 \to \mathbb{P}_3$ is given by $T(p(x)) = p'(x)$. Find a matrix M which represents T relative to the standard basis.
- Suppose that $T\colon M_{2\times 2} \to \mathbb{R}$ is given by $T(A) = \operatorname{tr}(A + A^t)$. (Here tr represents the *trace*, meaning the sum of the diagonal entries.)
  Show that T is a linear transformation. Then find a matrix which represents T relative to the standard basis for $M_{2\times 2}$.

(3) Suppose we have two different basis $\mathcal{B}_1$ and $\mathcal{B}_2$ for vector space V. Explain how to construct a matrix which converts between coordinates relative to the two bases.

- Give two different bases for $\mathbb{P}_3$ and find the matrix S which converts between coordinates relative to the two bases.
- Give two different bases for $M_{2\times 2}$ and find the matrix S which converts between coordinates relative to the two bases.

(4) Suppose we have a transformation $T\colon V \to V$. What does it mean to say that $\mathcal{E}$ is an eigenbasis for V relative to transformation T.

- Let $H\colon \mathbb{P}_3 \to \mathbb{P}_3$ be given by
  $$H(p(x)) = e^{x^2} \frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}[p(x)]\right].$$
  Show that H is indeed a linear transformation. Then find an eigenbasis for $\mathbb{P}_3$ relative to H. What matrix converts between coordinates relative to the eigenbasis and coordinates relative to the standard basis? What matrix represents H relative to coordinates relative to the eigenbasis?
- Let $T\colon M_{2\times 2} \to M_{2\times 2}$ be given by
  $$T(M) = \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} M.$$
  Find a eigenbasis for $M_{2\times 2}$ relative to T. What matrix converts between coordinates relative to the eigenbasis and coordinates relative to the standard basis? What matrix represents T relative to coordinates relative to the eigenbasis?

Exercise 19.10. Understand the dot product on $\mathbb{R}^n$.

(1) Give the definition of the dot product in terms of transpose and multiplication. What key properties does the dot product have?
(2) Define the following terms: Orthogonal, orthonormal, orthogonal complement, projection.
(3) Explain how to compute $\text{proj}_{\text{span}\{\mathbf{x}\}}\,\mathbf{u}$, and explain geometrically what it means. Then provide an illustrative example.
(4) Explain the procedure for constructing an orthonormal basis for a subspace V of $\mathbb{R}^n$. Illustrate your procedure by finding an orthonormal basis for

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

(5) Explain what is meant by $\text{proj}_V\,\mathbf{u}$ (where V is some subspace). Your explanation should include a geometric description as well as a method for computing the projection.
(6) Compute the projection of $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on to the subspace V above.
(7) Explain what is meant by $\text{proj}_{V^\perp}\,\mathbf{u}$. Your explanation should include a geometric description as well as a method for computing the projection.
(8) Compute the projection of $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on to $V^\perp$, where V is as above.
(9) Explain the connection between symmetric matrices and orthogonal eigenspaces. Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 5 \end{pmatrix}.$$

Exercise 19.11. Understand inner products on various vector spaces.

(1) Give the definition of an inner product.
(2) Here we explore an alternate inner product on $\mathbb{R}^n$:
   - Suppose A is a $n \times n$ matrix which is symmetric and has positive eigenvalues. I claim that A can be used to define an inner product on $\mathbb{R}^n$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle_A = \mathbf{u}^t A \mathbf{v}.$$

Show that this satisfies the symmetric and linear properties of an inner product; you do not need to prove the positive definite property (but it is true!).

- Now restrict attention to $\mathbb{R}^3$ and the matrix
$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
Compute
$$\langle \mathbf{u}, \mathbf{v} \rangle_A \qquad \text{where } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$
- With A as above, find a basis for $\mathbb{R}^3$ which is orthonormal with respect to the inner product determined by A.

(3) Here we explore the standard $\langle \, , \, \rangle_P$ inner product on $\mathbb{P}_3$.
- First, give the definition of this standard inner product.
- Let $p(x) = x$ and $q(x) = 1 + x + x^2$. Compute $\text{proj}_{\text{span}\{p(x)\}} q(x)$ and $\text{proj}_{\text{span}\{p(x)\}^\perp} q(x)$.
- Find an orthonormal basis for $\text{span}\{p(x)\}^\perp$.
- Use the orthonormal basis for $\text{span}\{p(x)\}^\perp$ to construct a related orthonormal basis for $\mathbb{P}_3$.

(4) Here we explore a "modified L^2 norm for polynomials:
- We work with $\mathbb{P}_3$ and define
$$\langle p(x), q(x) \rangle = \int_{-\infty}^{\infty} e^{-x^2} \, dx.$$
Explain why this is a valid inner product.
Notice that it is a real pain to actually compute this inner product in most cases. . . nevertheless, show that 1 and x are orthogonal, but that 1 and x^2 are not.
- Let $H \colon \mathbb{P}_3 \to \mathbb{P}_3$ be given by
$$H(p(x)) = e^{x^2} \frac{d}{dx} \left[e^{-x^2} \frac{d}{dx} [p(x)] \right].$$
Show that this transformation is self-adjoint ("symmetric").

- Explain how we know that an eigenbasis for $\mathbb{P}_3$ (relative to H) must be orthogonal. Notice that you constructed an eigenbasis you earlier!

(5) Finally, we discuss an inner product for $M_{2\times2}$.

- Let matrix M be given by

$$M = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

  Define $\langle A, B\rangle_M = \operatorname{tr}(A^t M B)$. Explain why this is an inner product. (Here tr represents the *trace* of the matrix, meaning the sum of the diagonal elements.)

- Find the inner product of all pairs of standard basis vectors of $M_{2\times2}$. Use this to construct a matrix which, in coordinates given by the standard basis, represents the inner product.
- Consider now the transformation $T\colon M_{2\times2} \to M_{2\times2}$ given by $T(A) = MA$. Find a basis of $M_{2\times2}$ consisting of eigenmatrices of T.
- I claim that if A_1 and A_2 are two eigenmatrices of T with different eigenvalues, then $\langle A, B\rangle_M = 0$. Justify this claim.
- Use the above to construct a basis for $M_{2\times2}$ which is orthonormal relative to the inner product $\langle\cdot,\cdot\rangle_M$.