TOPIC 18

Inner product spaces

Exercise 18.1. In this problem we investigate the inner product

$$\langle p, q\rangle_P = \int_{-1}^{1} p(x)\, q(x)\, dx.$$

You are welcome to use technology in order to compute various integrals... but need to be clear about which integrals you are having the computer evaluate.

(1) Use the inner product to find the norm of the polynomial $r(x) = 1 + x + x^2 + x^3$. Then find the angle between the polynomial $e_1 = x$ and $e_2 = x^2$.
(2) Show that the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$ is not an orthogonal basis for $\mathbb{P}_3$. Use the Gram-Schmidt method to find an orthonormal basis for $\mathbb{P}_3$.
(3) What is the coordinate expression of $r(x)$ relative to your orthonormal basis?
(4) Find a basis for the orthogonal complement to span$\{x\}$ in $\mathbb{P}_3$. (That is, find a basis for the vector space of all polynomials in $\mathbb{P}_3$ that are orthogonal to x.
(5) The vector space $\mathbb{P}_3$ is a subspace of the vector space of all continuous functions. Explain how to use the orthogonal basis that you constructed above in order to find the projection of a function $f(x)$ on to $\mathbb{P}_3$. Illustrate your method by finding

$$\mathrm{proj}_{\mathbb{P}_3}(\sin x).$$

Exercise 18.2. In this problem we study linear transformations of the vector space $\mathbb{P}_3$ using the inner product appearing in the previous problem.

(1) What is the adjoint of a linear transformation? Use integration by parts to show that if $T(p) = (1 - x^2)\frac{dp}{dx}$ then $T^*(p) = -(1 - x^2)\frac{dp}{dx} + 2xp$.
(2) Show that the linear operator

$$F(p) = (1 - x^2)\frac{d^2p}{dx^2} - 2x\frac{dp}{dx}$$

is self-adjoint with respect to this inner product.
(3) Explain why the eigenspaces of a self-adjoint linear transformation must be orthogonal.
(4) Find an orthonormal basis of $\mathbb{P}_3$ consisting of eigenpolynomials of F.

Exercise 18.3. In this exercise we continue to explore $\mathbb{P}_3$ with the inner product above.

(1) Let $\mathcal{E} = \{e_0, e_1, e_2, e_3\}$ be the standard basis. Compute the following numbers

$$g_{00} = \langle e_0, e_0\rangle_P \qquad g_{01} = g_{10} = \langle e_0, e_1\rangle_P \qquad g_{02} = g_{20} = \langle e_0, e_2\rangle_P$$
$$g_{03} = g_{30} = \langle e_0, e_3\rangle_P \qquad g_{11} = \langle e_1, e_1\rangle_P \qquad g_{12} = g_{21} = \langle e_1, e_2\rangle_P$$
$$g_{13} = g_{31} = \langle e_1, e_3\rangle_P \qquad g_{22} = \langle e_2, e_2\rangle_P$$
$$g_{23} = g_{32} = \langle e_2, e_3\rangle_P \qquad g_{33} = \langle e_3, e_3\rangle_P$$

and arrange them in to a matrix

$$G_{\mathcal{E}} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

(2) Suppose that we have polynomials p and q such that

$$[p]_{\mathcal{E}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad \text{and} \quad [q]_{\mathcal{E}} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Show that

$$\langle p, q\rangle_P = [p]_{\mathcal{E}}^t \, G \, [q]_{\mathcal{E}}$$

and thus that G represents the inner product relative to the standard basis $\mathcal{E}$.

(3) Use the matrix G to compute $\langle 1 - x - x^2 - x^3, 1 + 2x + 3x^2 + 4x^3\rangle_P$. Verify your answer by computing the inner product using integration.

(4) What matrix represents the inner product relative to an orthonormal basis? Explain your reasoning.