TOPIC 17

Applications of the dot product

Exercise 17.1. In this problem we study the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given relative to the standard coordinates by

$$T(\mathbf{x}) = M\mathbf{x} \quad \text{with} \quad M = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

(1) What does it mean to say that the matrix A is symmetric?
(2) What does it mean to say that T is self-adjoint? Explain the connection between T being self-adjoint and A being symmetric.
(3) Explain why symmetric matrices must have orthogonal eigenspaces.
(4) Find the eigenspaces of T and verify they are orthogonal.
(5) Construct an orthonormal basis, called $\mathcal{B}$, for $\mathbb{R}^4$ consisting of eigenvectors.
(6) What does it mean for a matrix to be orthogonal? Explain why the matrix $S_{\mathcal{E}\mathcal{B}}$ that converts between basis $\mathcal{B}$ and the standard basis $\mathcal{E}$ is an orthogonal matrix. What is $S_{\mathcal{E}\mathcal{B}}^{-1}$?
(7) Find the matrix $M_{\mathcal{B}}$ which represents T in coordinates determined by $\mathcal{B}$.
(8) Consider the vectors

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 3 \\ -1 \end{pmatrix}$$

(a) Verify that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
(b) Find the expressions of $\mathbf{w}$ and $\mathbf{v}$ in coordinates determined by basis $\mathcal{B}$. Verify that these coordinate expressions are still orthogonal with respect to the usual dot product.
(c) Find $[T(\mathbf{v})]_{\mathcal{B}}$ and $[T(\mathbf{w})]_{\mathcal{B}}$.

Exercise 17.2. Suppose we have the following points in the plane:

$$(1, 2), \quad (1, 3), \quad (2, 3), \quad (4, 3), \quad (3, 4), \quad (2, 1).$$

(1) Find the matrix M and vector $\mathbf{y}$ that describe the least-squares line-of-best-fit problem. Then solve the system (as described in class) to find the line of best fit. $y = m_* + b_*$.

(2) Make a careful plot of the points above as well as the line of best fit.

(3) Let $\bar{x}$ be the average of the x values and $\bar{y}$ be the average of the y values. Show that the point $(\bar{x}, \bar{y})$ lies on the line of best fit.

(4) (Optional challenge) The general least square problem seeks the best approximation of the system

$$\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Use the procedure in class to find the best approximate solution m_*, b_*. You should get a formula that involves the quantities

$$\overline{x} = \frac{1}{n}\sum_i x_i \qquad \overline{y} = \frac{1}{n}\sum_i y_i \qquad \overline{x^2} = \frac{1}{n}\sum_i (x_i)^2 \qquad \overline{xy} = \frac{1}{n}\sum_i x_i y_i$$

After you are done, check out https://en.wikipedia.org/wiki/Ordinary_least_squares

Exercise 17.3. (Optional fun) One can generalize least-squares problem to curve-fitting as well as to higher dimensional data. Read through section 4.4 in *Linear Algebra Done Wrong* (a link is available on the course website) and then do Exercise 4.4 on page 141.