TOPIC 16

The dot product

Lecture

Basic concepts

- Definition of dot product

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$$

- Norm of a vector $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$
- Law of Cosines, Cauchy-Schwartz

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$
$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Projections

- Consider line $\Lambda = \text{span}\{\mathbf{v}\}$
- Definition of $\text{proj}_\Lambda : \mathbb{R}^n \to \mathbb{R}^n$:

$$\text{proj}_\Lambda(\mathbf{w}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

- Projection is a linear transformation: check definition
- Example in $\mathbb{R}^2$
- Kernel of proj_Λ is $\Lambda^\perp$, the *orthogonal complement* of Λ
- Decomposition of vectors: Any vector $\mathbf{w}$ can be written uniquely as

$$\mathbf{w} = \text{proj}_\Lambda(\mathbf{w}) + \mathbf{w}^\perp$$

- We can define projection $\text{proj}_{\Lambda^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\text{proj}_{\Lambda^\perp}(\mathbf{w}) = \mathbf{w} - \text{proj}_\Lambda(\mathbf{w})$$

Dot product and transpose

- Notice that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^t \mathbf{w}$.
- Helpful fact: $(AB)^t = B^t A^t$

Adjoint transformations

- Suppose we have $f : \mathbb{R}^n \to \mathbb{R}^m$, given in standard basis by matrix M. Note that M is an $m \times m$ matrix.
- Let $\mathbf{v}$ be in $\mathbb{R}^n$ and $\mathbf{w}$ be in $\mathbb{R}^m$. We compute

$$(M\mathbf{v}) \cdot \mathbf{w} = v \cdot (M^t \mathbf{w}).$$

- We define $f^* : \mathbb{R}^m \to \mathbb{R}^n$ by $f^*(\gtrapprox) = M^t \gtrapprox$. This is called the *adjoint transformation of* f.
- We have $f(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot f^*(\mathbf{w})$.
- Key fact about fundamental spaces:

$$\begin{aligned} \operatorname{ran}(f)^\perp = \ker(f^*) \quad &\leftrightarrow \quad \operatorname{ran}(f) = \ker(f^*)^\perp \\ \operatorname{ran}(f^*)^\perp = \ker(f) \quad &\leftrightarrow \quad \operatorname{ran}(f^*) = \ker(f)^\perp \end{aligned}$$

Self-adjoint transformations

- If $f : \mathbb{R}^n \to \mathbb{R}^n$, then $f^* : \mathbb{R}^n \to \mathbb{R}^n$.
- If $f = f^*$ we say that f is *self-adjoint*.
- Self-adjoint transformations of $\mathbb{R}^n$ are given in standard coordinates by symmetric matrices
- Key fact: self-adjoint transformations have orthogonal eigenspaces

Activity: Orthogonality

(1) Recall that two vectors are called *orthogonal* if their dot product is zero. We now extend this definition:

- A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is orthogonal if ____.
- Suppose V is a subspace of $\mathbb{R}^n$. An *orthogonal basis* of V is ____.

(2) We can use projections to construct orthogonal bases. For example, suppose we have a plane with basis $\{\mathbf{b}_1, \mathbf{b}_2\}$.

(a) Let $\Lambda_1 = \text{span}\{\mathbf{b}_1\}$. Explain why we know (even before computing) both $\text{proj}_{\Lambda_1}(\mathbf{b}_2)$ and $\text{proj}_{\Lambda_1^\perp}(\mathbf{b}_2)$ are also in the plane.

(b) Explain why $\{\mathbf{b}_1, \text{proj}_{\Lambda_1^\perp}(\mathbf{b}_2)\}$ form an orthogonal basis for the plane.

(c) Find an orthogonal basis for the plane $x - 2y + 4z = 0$.

(3) We now make another definition. We say that a list of vectors are *orthonormal* if they are orthogonal and if the norm of each vector is 1. The motivation behind this definition is that such collections of vectors behave like the standard basis vectors.

(a) An *orthonormal basis* is ____.

(b) Explain how to take an orthogonal basis and construct an orthonormal basis.

(c) Find an orthonormal basis for the plane $x - 2y + 4z = 0$.

(4) Suppose we have a three-dimensional subspace with basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Construct an iterative procedure by which one can use projections to construct an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ for the subspace.

Note: This process is called the Gram-Schmidt *process.*

Homework

Exercise 16.1. In class I interpreted the Law of Cosines as a generalization of the Pythagorean theorem. Here we explore the Pythagorean theorem from a new "reversed" perspective.

(1) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$. Show that $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.
(2) Give an example of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ which are not orthogonal. Is it the case that $||\mathbf{a} + \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$?
(3) Explain why $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ holds if and only if the vectors are orthogonal.

Exercise 16.2. In this problem, we suppose $\mathcal{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Thus any vector $\mathbf{v}$ can be written

$$\mathbf{v} = v_1\mathbf{f}_1 + v_2\mathbf{f}_2 + \cdots + v_n\mathbf{f}_n,$$

where v_k are constants.

(1) Show that $v_k = \mathbf{v} \cdot \mathbf{f}_k$.
(2) Show that
$$||\mathbf{v}||^2 = \sum_{k=1}^{n} (v_k)^2.$$
(3) Interpret the previous results in the language of coordinates determined by the basis $\mathcal{U}$ by completing the following:
   - If $\mathcal{F}$ is an orthonormal basis and then in order to find $[\mathbf{v}]_{\mathcal{F}}$ we can ...
   - If $\mathcal{F}$ is an orthonormal basis and $[\mathbf{v}]_{\mathcal{F}} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ then $||\mathbf{v}|| =$...
(4) In what way do orthonormal bases "behave like the standard basis"?

Exercise 16.3. In class we defined the projection operator proj_{Λ} for a line Λ. The goal of this exercise is to think about how to extend this to projections on to other subspaces. To to this, we consider the subspace $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this problem we also use the vector

$$\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

(1) What should be the definition of $V^{\perp}$?

(2) Give a geometric description (in words, without formulas!) for what $\text{proj}_V(\mathbf{w})$ and $\text{proj}_{V^{\perp}}(\mathbf{w})$ should mean. Explain why it should be the case that

$$\mathbf{w} = \text{proj}_V(\mathbf{w}) + \text{proj}\, V^{\perp}(\mathbf{w}).$$

(3) Compute $\text{proj}_{\text{span}\{\mathbf{v}_1\}}(\mathbf{w})$ and $\text{proj}_{\text{span}\{\mathbf{v}_2\}}(\mathbf{w})$.

(4) Find an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ for V.

(5) Compute $\text{proj}_{\text{span}\{\mathbf{f}_1\}}(\mathbf{w})$ and $\text{proj}_{\text{span}\{\mathbf{f}_2\}}(\mathbf{w})$.

(6) Show that

$$\text{proj}_{\text{span}\{\mathbf{v}_1\}}(\mathbf{w}) + \text{proj}_{\text{span}\{\mathbf{v}_2\}}(\mathbf{w})$$

and

$$\text{proj}_{\text{span}\{\mathbf{f}_1\}}(\mathbf{w}) + \text{proj}_{\text{span}\{\mathbf{f}_2\}}(\mathbf{w})$$

are not the same. Explain what is happening geometrically.

(7) I claim that we should define

$$\text{proj}_V(\mathbf{w}) = \text{proj}_{\text{span}\{\mathbf{f}_1\}}(\mathbf{w}) + \text{proj}_{\text{span}\{\mathbf{f}_2\}}(\mathbf{w}).$$

Explain why this is a reasonable definition.

(8) What should be the definition of $\text{proj}_{V^{\perp}}$? Compute $\text{proj}_{V^{\perp}}(\mathbf{w})$.

(9) Find an orthonormal basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ for $\mathbb{R}^4$, where $\mathbf{f}_1$ and $\mathbf{f}_2$ are the same vectors you found above.

(10) Express $\mathbf{w}$ in coordinates with respect to the basis $\mathcal{F}$.

(11) Give the relationship between the coordinate representation of $\mathbf{w}$ relative to $\mathcal{F}$ and the projections computed earlier.

(12) Use the coordinate expression $[\mathbf{w}]_{\mathcal{F}}$ in order to compute $\|\mathbf{w}\|$. Compare with the "usual" way of finding the norm.