TOPIC 15

More abstract linear spaces

Exercise 15.1. Suppose V and W are real vector spaces. Give definitions of...

(1) ... what it means for a collection $v_1, \dots, v_n \in V$ to be *linearly independent*.
(2) ... what it means for $\{v_1, \dots, v_n\}$ to be a *basis* for V.
(3) ... what it means for the *dimension of V to be n*.
(4) ... what it means to express $w \in V$ in terms of the coordinates determined by basis $\mathcal{B}$.
(5) ... what it means for $T\colon V \to W$ to be a linear transformation.
(6) ... what the kernel and image of transformation $T\colon V \to W$ are.
(7) ... what it means for matrix $M_{\mathcal{B}}$ to represent a transformation $T\colon V \to W$ in coordinates relative to basis $\mathcal{B}$.

Exercise 15.2. In this problem we consider the vector space $M_{2\times 2}$ of 2×2 matrices. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be the standard basis for $M_{2\times 2}$.

(1) Show that

$$C = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

also forms a basis for $M_{2\times 2}$.

(2) Express the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

in coordinates relative to basis $\mathcal{B}$ and in coordinates relative to basis C.

(3) Find a matrix S which can be used to convert between coordinate expressions relative to basis $\mathcal{B}$ and coordinate expressions relative to basis C.

(4) Consider the linear transformation $T\colon M_{2\times 2} \to M_{2\times 2}$ given by

$$T(M) = \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix} M.$$

(a) Find a matrix $M_{\mathcal{B}}$ which describes T in coordinates relative to basis $\mathcal{B}$.
(b) Find a matrix M_{C} which describes T in coordinates relative to basis C.
(c) Find the eigenvalues and eigenmatrices of the transformation T.
(d) Find a basis of $M_{2\times 2}$ consisting of eigenmatrices of the transformation T. What is the matrix representing T in this basis? What matrix converts between coordinates relative to this basis and coordinates relative to the basis $\mathcal{B}$?

(5) Let $S_{2\times 2}$ represent the symmetric 2×2 matrices. Consider the linear transformation $P : M_{2\times 2} \to S_{2\times 2}$ given by

$$P(M) = \frac{1}{2}\left(M + M^t\right).$$

(a) Verify that P is indeed a linear transformation and that $P(M)$ is symmetric for all $M \in M_{2\times 2}$.
(b) Construct a "standard" basis $\mathcal{F}$ for $S_{2\times 2}$. What is the dimension of $S_{2\times 2}$?
(c) Find the matrix describing P relative to coordinates given by $\mathcal{B}$ and $\mathcal{F}$.
(d) Give a careful description of $\ker P$ and $\operatorname{im} P$.
(e) Give a careful description of all matrices M satisfying

$$P(M) = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

How many such matrices are not invertible?

Exercise 15.3. Here we consider two linear transformations

$$\mathcal{A} : \mathbb{P}_3 \to \mathbb{P}_4$$
$$\mathcal{D} : \mathbb{P}_4 \to \mathbb{P}_3$$

given by

$$\mathcal{A}[p] = \int_1^x p(y)\,dy \quad \text{and} \quad \mathcal{D}[p] = \frac{d}{dx}\,[p(x)]$$

(1) Verify that both $\mathcal{A}$ and $\mathcal{D}$ are linear transformations from the spaces indicated above.
(2) Consider the standard bases

$$\mathcal{E}_4 = \{1, x, x^2, x^3\}$$
$$\mathcal{E}_5 = \{1, x, x^2, x^3, x^4\}$$

of $\mathbb{P}_3$ and $\mathbb{P}_4$ respectively. Construct a matrix A which describes $\mathcal{A}$ relative to these bases, and a matrix D which describes $\mathcal{D}$ relative to these bases.
(3) Find the kernel and image of both $\mathcal{A}$ and $\mathcal{D}$.

(4) We now consider the composition $\mathcal{D} \circ \mathcal{A} \colon \mathbb{P}_3 \to \mathbb{P}_3$ defined by $p \mapsto \mathcal{D}(\mathcal{A}(p))$. Show that this map is the *identity map*, meaning the map which takes an element to itself, and is given (relative to coordinates determined using $\mathcal{B}_4$) by the identity matrix.

(5) State the First Fundamental Theorem of Calculus. How does this relate to the previous part of this exercise?