TOPIC 14

The vector space $\mathbb{P}_3$

For this assignment we focus on the vector space $\mathbb{P}_3$, the space of all polynomials of degree less than 4 (ie, less than or equal to three). We make use of the following bases:

$$\mathcal{A} = \{a_0 = 1, \quad a_1 = x, \quad a_2 = \frac{1}{2}x^2, \quad a_3 = \frac{1}{6}x^3\}$$
$$\mathcal{E} = \{e_0 = 1, \quad e_1 = x, \quad e_2 = x^2, \quad e_3 = x^3\}$$
$$C = \{c_0 = 1, \quad c_1 = 1 + x, \quad c_2 = 1 + x + x^2, \quad c_3 = 1 + x + x^2 + x^3\}$$

Exercise 14.1. Express the polynomials $p_c = 1 - \frac{1}{2}x^2$ and $p_s = x - \frac{1}{6}x^3$ in terms of each of the three bases $\mathcal{A}, \mathcal{E}, C$

Exercise 14.2.

(1) Find the matrix $S_{\mathcal{E}C}$ which converts between bases C and $\mathcal{E}$.
(2) Find the matrix $S_{\mathcal{E}\mathcal{A}}$ which converts between bases $\mathcal{A}$ and $\mathcal{E}$.
(3) Use the previous matrices to build a matrix $S_{C\mathcal{A}}$ which converts between bases $\mathcal{A}$ and C.

Exercise 14.3. Express the polynomial $p_e = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ in terms of the basis $\mathcal{A}$. Then use the matrices you found above to express p_e relative to the other two bases.

Exercise 14.4. Let S_0 be the collection of all polynomials p in $\mathbb{P}_3$ such that $p(1) = 0$. Is S_0 a vector space?

Exercise 14.5. Let S_1 be the collection of all polynomials p in $\mathbb{P}_3$ such that $p(1) = 1$. Is S_1 a vector space?

Exercise 14.6. Let J be the collection of all polynomials p in $\mathbb{P}_3$ such that $\int_0^1 p(x)dx = 0$. Is J a vector space?

Exercise 14.7. Here we study the linear transformation $L\colon \mathbb{P}_3 \to \mathbb{P}_3$ given by

$$(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

(1) Verify that L is indeed a linear transformation $\mathbb{P}_3 \to \mathbb{P}_3$
(2) Find a matrix $M_{\mathcal{E}}$ which expresses L relative to the standard basis $\mathcal{E}$.
(3) Find a basis for the null space of $M_{\mathcal{E}}$. Use this to find a basis for $\ker(L)$.
(4) Find a basis for the columnspace of $M_{\mathcal{E}}$. Use this to find a basis for $\operatorname{im}(L)$.
(5) Verify that the Rank-Nullity theorem holds for the transformation L.
(6) Let $q = 1+2x+3x^2+4x^3$. Describe the space of all solutions p to the equation $L(p) = q$.
(7) Let $r = 1+2x-3x^2-4x^3$. Describe the space of all solutions p to the equation $L(p) = r$.
(8) Find the eigenvalues and eigenspaces of $M_{\mathcal{E}}$. Use these to determine the eigenvalues and eigenspaces of L.
(9) Construct a basis for $\mathbb{P}_3$ consisting of eigen-polynomials p of L which have the property that $p(1) = 1$. Call this basis $\mathcal{F}$.
(10) Construct a matrix S which converts between the basis $\mathcal{E}$ to basis $\mathcal{F}$.
(11) Use the matrix S to convert the matrix $M_{\mathcal{E}}$ to the matrix $M_{\mathcal{F}}$ which describes L relative to coordinates determined by $\mathcal{F}$.
(12) Read the article http://en.wikipedia.org/wiki/Adrien-Marie_Legendre