TOPIC 13

Abstract vector spaces

Basic ideas

- Vector space is a collection of objects that "behave" like vectors in $\mathbb{R}^n$
- More specific, a vector space is a collection of objects V, a notion of addition $\oplus$, and a notion of multiplying by scalars $\odot$ that obeys the usual distributive and associative rules...
- Most of the time we use the "usual" addition and multiplication, and thus don't bother with circling the operations

Examples

- differentiable function
- matrices (of fixed size)
- sequences

Finite dimensional case

- In this class we focus on vector spaces where there is a finite-dimensional basis
- Definition of basis, span, etc. are all the same as in $\mathbb{R}^n$
- Using a basis we can express elements in coordinates, thus there is a correspondence between the vector space and $\mathbb{R}^n$

Example: $\mathbb{P}_3$, collection of polynomials of degree less than or equal to three

- Standard basis $\mathcal{E} = \{e_0 = 1, e_1 = x, e_2 = x^2, e_3 = x^4\}$

- If $p = 2x^2 - 7x^3$ we have

$$[p]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -7 \end{bmatrix}$$

- $f(p) = \frac{d}{dx}[xp]$ is a linear transformation $\mathbb{P}_3 \to \mathbb{P}_3$.

Exercise 13.1. Let V be the set of positive real numbers and consider the operation $\oplus$ determined by

$$x \oplus y = xy.$$

For a real number α and $x \in V$ define the operation

$$\alpha \odot x = x^{\alpha}.$$

Verify that V is a real vector space with this version of "addition" and "multiplication by real numbers."

Exercise 13.2. Show that all functions $f(x)$ that satisfy

$$\frac{d^2}{dx^2}[f] + 3e^x \frac{d}{dx}[f] = 0$$

form a vector space.