TOPIC 12

Coordinates

Motivating example:

- Consider two bases for $\mathbb{R}^2$
  $$\mathcal{E} = \left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$
- Given a vector $\mathbf{v}$ we can express $\mathbf{v}$ relative to $\mathcal{E}$ and/or relative to $\mathcal{B}$.
- For example, consider $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. We have
  $$\begin{aligned} \mathbf{v} &= 3\mathbf{e}_1 + 7\mathbf{e}_2 \\ &= 5\mathbf{b}_1 + (-2)\mathbf{b}_2. \end{aligned}$$
- We write this as
  $$[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$
- We say that these are the *expressions of* $\mathbf{v}$ *in coordinates* relative to the bases $\mathcal{E}$ and $\mathcal{B}$.
- Look at the picture. . .

General situation

- The "usual expression" of a vector is with respect to the standard basis:
  $$\left[\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \right]_{\mathcal{E}} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$
- We use square brackets for general coordinates, round brackets for Cartesian coordinates. If no basis is specified, assume the standard basis.
- We write $\mathbb{R}^n_{\mathcal{B}}$ to indicate we are using basis $\mathcal{B}$.

- We can express linear transformations relative to bases.
- Many times we want to use coordinates that are well-adapted to the situation/problem at hand. . .
- We can view change of basis as a linear transformation — be careful, now vectors do not represent displacements in the same way!

Example

- Convert between $\mathcal{E}$ and $\mathcal{B}$ above
- We want a linear transformation that takes $[\mathbf{v}]_{\mathcal{B}} \mapsto [\mathbf{v}]_{\mathcal{E}}$. We have
$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} \mapsto \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}} \qquad \begin{bmatrix}0\\1\end{bmatrix}_{\mathcal{B}} \mapsto \begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{E}}$$
- Thus multiplying by the matrix
$$S = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \tag{2}$$
does conversion, in sense that $[\mathbf{v}]_{\mathcal{E}} = S\,[\mathbf{v}]_{\mathcal{B}}$.
- Reverse conversion is done by
$$S^{-1} = \begin{pmatrix}1/2 & 1/2\\1/2 & -1/2\end{pmatrix}.$$
- Examples. . .

Application to linear transformations (example)

- Have transformation $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x-5y\\-5x+y\end{pmatrix}$
- Relative to standard basis, we have
$$f(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2) = (\alpha_1 - 5\alpha_2)\mathbf{e}_1 + (-5\alpha_1 + \alpha_2)\mathbf{e}_2$$
and f is given by matrix $A_{\mathcal{E}} = \begin{pmatrix}1 & -5\\-5 & 1\end{pmatrix}$

- Let's find an *eigenbasis*, meaning a basis of eigenvectors. Compute eigenstuff:

$$\lambda = -4 \quad \rightsquigarrow \quad \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

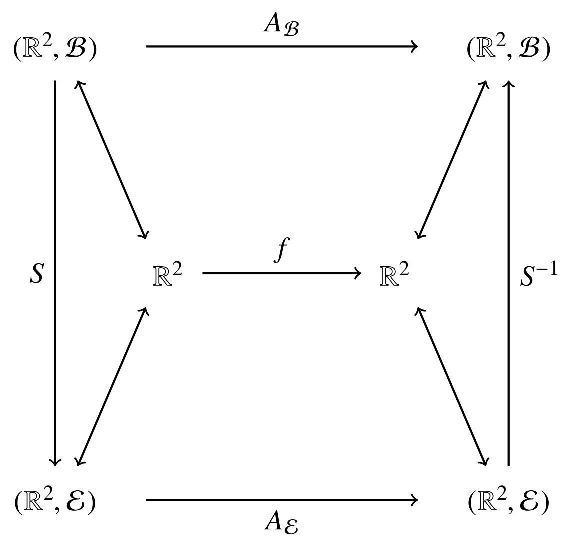
$$\lambda = 6 \quad \rightsquigarrow \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Relative to the eigenbasis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ we have

$$f(\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2) = -4\beta_1\mathbf{b}_1 + 6\beta_2\mathbf{b}_2$$

and f is given by the matrix $A_{\mathcal{B}} = \begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix}$ Much simpler!

- Draw the big diagram. . . and chase vector around!



- Thus $A_{\mathcal{B}} = S^{-1}A_{\mathcal{E}}S$ and $A_{\mathcal{E}} = SA_{\mathcal{B}}S^{-1}$
- Relative to an eigenbasis, the matrix that represents a transformation is *diagonal*. Thus we have

expressing a transformation relative to an eigenbasis $\quad\leftrightarrow\quad$ diagonalization a matrix

Exercise 12.1. Consider the collection of vectors

$$\mathcal{F} = \left\{ \mathbf{f}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{f}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \right\}$$

(1) Show that $\mathcal{F}$ is a basis for $\mathbb{R}^3$.
(2) Construct a matrix $S_{\mathcal{E},\mathcal{F}}$ such that $[\mathbf{v}]_{\mathcal{E}} = S_{\mathcal{E},\mathcal{F}}[\mathbf{v}]_{\mathcal{F}}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^3$
(3) Use this matrix to express $\mathbf{v} = 3\mathbf{e}_1 - 7\mathbf{e}_3$ in coordinates relative to $\mathcal{F}$.

Exercise 12.2. Consider the transformation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3x + 4y - z \\ 4y \\ -x + 4y - 3z \end{pmatrix}.$$

(1) Find the matrix $M_{\mathcal{E}}$ that represents f relative to the standard basis $\mathcal{E}$.
(2) Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ consisting of eigenvectors of f.
(3) Find the matrix that represents f relative to your eigenbasis $\mathcal{B}$.
(4) Find the matrix $S_{\mathcal{E},\mathcal{B}}$ such that $[\mathbf{v}]_{\mathcal{E}} = S_{\mathcal{E},\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^3$
(5) Use the matrix $S_{\mathcal{E},\mathcal{B}}$ and the results of Exercise 12.1 to find the matrix $M_{\mathcal{F}}$ that represents f relative to the basis $\mathcal{F}$ above.

Exercise 12.3. In this exercise we see how to use change-of-basis to engineer transformations. Our goal is to construct a transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

- The line $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$ is stretched by a factor of 5.
- There is some sort of rotation of the plane $x + y + z = 0$.

(1) Construct a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbb{R}^3$ such that $\mathbf{b}_1$ lies on the line above, and such that $\mathbf{b}_2, \mathbf{b}_3$ span the plane indicated above.
(2) Construct a matrix $S_{\mathcal{E},\mathcal{B}}$ such that $[\mathbf{v}]_{\mathcal{E}} = S_{\mathcal{E},\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^3$.
(3) Let f be the transformation satisfying

$$f(\mathbf{b}_1) = 5\mathbf{b}_1, \qquad f(\mathbf{b}_2) = \mathbf{b}_3, \qquad f(\mathbf{b}_3) = -\mathbf{b}_2.$$

Explain why this choice satisfies the criteria above.
Find the matrix $M_{\mathcal{B}}$ that represents f in coordinates determined by $\mathcal{B}$?.
(4) Use $M_{\mathcal{B}}$ and $S_{\mathcal{E},\mathcal{B}}$ to construct a matrix that represents f relative to the standard basis for $\mathbb{R}^3$. Then write down a formula for $f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.