TOPIC 11

Matrix multiplication

Idea

- In one dimension linear transformations take the form $f(x) = ax$.
- Can we generalize this to higher dimensions?

Example

- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 5y \\ 5x + y \end{pmatrix}$
- Thus
$$f(\mathbf{e}_1) = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \qquad f(\mathbf{e}_2) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
and
$$f\begin{pmatrix} x \\ y \end{pmatrix} = f(x\mathbf{e}_1) + f(y\mathbf{e}_2) = xf(\mathbf{e}_1) + yf(\mathbf{e}_2) = x\begin{pmatrix} 1 \\ -5 \end{pmatrix} + y\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
- We define
$$\begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ -5 \end{pmatrix} + y\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

More generally

- We define multiplication of a matrix and a vector by "take each column vector of matrix, multiply by corresponding entry of vector, then add it all up"

Example

- $\begin{pmatrix} 1 & 3 & 5 \\ 4 & 7 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 9 \\ -3 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 4 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 7 \end{pmatrix} - 3\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 81 \end{pmatrix}$

Remarks

- We say that A is a $m \times n$ matrix if it has m rows and n columns. Such a matrix corresponds to a transformation $\mathbb{R}^n \to \mathbb{R}^m$.
- We can also write systems of equations using matrices. They take the form $A\mathbf{v} = \mathbf{r}$

Linear transformations and matrices

- Each linear transformation can be described by a matrix.
- There is also corresponding vocabulary:

Linear transformation	Matrix
$f\colon \mathbb{R}^n \to \mathbb{R}^m$	A is $m \times n$
kernel of f	*null space* of A
range of f	*column space* of A
f^{-1} is the *inverse* of f	A^{-1} is the *inverse* of A
determinant of f	*determinant* of A
f has no inverse	A is *singular*

Composing transformations and matrix multiplication

- Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and
- Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $B = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$
- What matrix describes $g \circ f$?
- Compute
$$BA\begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} 2x+3y \\ x+y \end{pmatrix} = (2x+2y)\begin{pmatrix} 4 \\ 1 \end{pmatrix} + (x+y)\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \dots$$
- Conclude that we should multiply matrices as follows
$$\begin{pmatrix} 1 & 3 & 5 \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} * & 2 & * \\ * & 4 & * \\ * & 5 & * \end{pmatrix} = \begin{pmatrix} * & 39 & * \\ * & * & * \\ * & * & * \end{pmatrix}$$
- In general we can multiply matrices if the dimensions match:
$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^p$$
works if A is $m \times n$ and B is $p \times m$. The product BA is $p \times n$.

- We can also add matrices if they are the same size. What does this mean in terms of transformations?

Determinants and matrices

- Geometrically, it must be that

$$\det(AB) = \det(A)\det(B)$$

and that

$$\det(A^{-1}) = \frac{1}{\det(M)}$$

Elementary matrices

- For each type of row operation there is an *elementary matrix* E such that multiplying on the left by E has the effect of the row operation.
- If a matrix M can be row reduced to the identity then it is the product of elementary matrices.
- This gives us a description of all invertible matrices!
- Any square matrix that is invertible can be written as the product of elementary matrices

Exercise 11.1. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For each of the following: either compute the quantity indicated or determine that the quantity does not consist of valid operations.

(1) $2A - 3B$
(2) $A + B + C$
(3) AB
(4) BA
(5) AC
(6) CA
(7) BC
(8) CB

Exercise 11.2. Express the following systems in matrix form (That is, write the system as $A\mathbf{v} = \mathbf{r}$ for some matrix A and vectors $\mathbf{r}, \mathbf{r}$.) *Do not solve the systems.*

(1) The system

$$\begin{aligned} x + y &= 3 \\ 2x - 3y &= 0 \end{aligned}$$

(2) The system

$$\begin{aligned} x + y + z &= 1 \\ x - z &= \pi \end{aligned}$$

(3) The system

$$\begin{aligned} x + y + z + w &= 3 \\ x + z &= 2 \\ y - w &= 9 \end{aligned}$$

Exercise 11.3. Suppose

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute A^n for every positive integer n.

Exercise 11.4. Find a 2×2 matrix B such that $B \neq 0$, but $B^2 = 0$. (When I write "$= 0$" here, what do I mean?)

Exercise 11.5.

(1) What are the three types of elementary matrices?
(2) What special properties do elementary matrices have?
(3) For each type of elementary matrix, answer the question: What happens to matrix A if we multiply it by the elementary matrix?
(4) How do the elementary matrices relate to our algorithm for finding matrix inverses?

Exercise 11.6. Compute (by hand, showing details) the inverses of

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 11.7. Suppose A and B are non-singular $n \times n$ matrices.

(1) How do we know that AB is non-singular?
(2) How is $(AB)^{-1}$ related to A^{-1} and B^{-1}?

Exercise 11.8.

(1) Figure out how to get *Mathematica* (or some other appropriate technology) to perform matrix operations, including: adding/subtracting, multiplication, inversion, transposition.
(2) Use technology to compute the square and inverse of

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 7 & -7 \end{pmatrix}$$

For more practice, I recommend Section MM of Beezer's notes (http://linear.ups.edu/html/section-MM.html) and/or Exercises 3.1, 3.2, 3.3, 3.4 of Linear Algebra Done Wrong (http://www.math.brown.edu/~treil/papers/LADW/LADW.html).