TOPIC 10

# Comments about eigenstuff

Characteristic polynomial

- Given a matrix $A$ we form the *characteristic polynomial* by

$$p(\lambda) = \det(A - \lambda I).$$

- The roots of the polynomial are the eigenvalues. Thus

$$p(\lambda) = (\#)(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_k)(\text{stuff that doesn't have any real roots})$$

- Sometimes we have repeated roots. The number of times a root appears is called the *algebraic multiplicity* of the root.

Examples:

- Consider the matrix

$$A_1 = \begin{pmatrix} -5 & 11 & 4 \\ 12 & -6 & -6 \\ -10 & 10 & 8 \end{pmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = -\lambda^3 - 3\lambda^2 + 90\lambda - 216 = -(\lambda - 6)(\lambda - 3)(\lambda + 12).$$

Each of the eigenvalues has algebraic multiplicity 1

- Consider the matrix

$$A_2 = \begin{pmatrix} -2 & 4 & 2 \\ 4 & -2 & -2 \\ -4 & 4 & 4 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = -(\lambda - 2)^2(\lambda + 4).$$

Thus $\lambda = 2$ has algebraic multiplicity 2 while $\lambda = -4$ has algebraic multiplicity 1.

- Consider the matrix

$$A_3 = \begin{pmatrix} -5 & 11 & 7 \\ 9 & -3 & -3 \\ -10 & 10 & 11 \end{pmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = -(\lambda - 6)^2(\lambda + 9)$$

The eigenvalue $\lambda = -9$ has algebraic multiplicity 1 and the eigenvalue $\lambda = 6$ has algebraic multiplicity 2.

Eigenspaces

- For each eigenvalue there is a corresponding eigenspace of eigenvectors.
- The dimension of the eigenspace is the *geometric multiplicity* of the eigenvalue.
- "Most" of the time we expect that the geometric multiplicity is the same as the algebraic multiplicity. However, this does not always happen. . .

Examples

- For the matrix $A_1$ above all three eigenspaces have dimension 1 and the algebraic multiplicity is the same as the geometric multiplicity.
- Consider the matrix $A_2$ above. The eigenspace for $\lambda = -4$ is one-dimensional and is spanned by $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The eigenspace for $\lambda = 2$ is two-dimensional and is spanned by $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Thus in both cases the algebraic multiplicity is the same as the geometric multiplicity.
- For the matrix $A_3$ above we compute that:
  - The eigenspace for $\lambda = -9$ is one-dimensional and is spanned by $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Thus the algebraic and geometric multiplicities of $\lambda = -9$ are the same.

– The eigenspace for $\lambda = 6$ is one dimensional and is spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Thus the algebraic multiplicity of $\lambda = 6$ is two, but the geometric multiplicity is 1.

Eigenbases

- In the best of all worlds, we can find a basis of eigenvectors.
- For $A_1$ and $A_2$ this is easy to do, since there are three linearly independent eigenvectors present.
- What about $A_3$? There is a problem with $\lambda = 6$ that we need to investigate.
- The eigenvectors for $\lambda = 6$ are the kernel of the transformation with matrix

$$\begin{pmatrix} -5-6 & 11 & 7 \\ 9 & -3-6 & 3 \\ -10 & 10 & 11-6 \end{pmatrix} = \underbrace{\begin{pmatrix} -11 & 11 & 7 \\ 9 & -9 & -3 \\ -10 & 10 & 5 \end{pmatrix}} \quad \text{RREF} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is where we get our first(and only) eigenvector for $\lambda = 6$

- In order to obtain another vector to use for our basis we consider the transformation

$$\tilde{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11x + 11y + 7z \\ 9x - 9y - 3z \\ -10z + 10y + 5z \end{pmatrix}$$

and seek a vector such that $\tilde{f}(\mathbf{v})$ is in the kernel of $\tilde{f}$. Such a vector must satisfy $\tilde{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. One possible solution is $\begin{pmatrix} 1/3 \\ 0 \\ 2/3 \end{pmatrix}$.

- Thus the "next best thing to an eigenbasis" for matrix $A_3$ is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$